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A STUDY OF THE TRANSIENT CHARACTERISTICS OF NETWORKS

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A STUDY OF THE TRANSIENT CHARACTERISTICS OF NETWORKS

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SUMMARY

The purpose of this investigation was to study the magnitude, phase and unit step response of low-pass network functions and to illustrate a technique using a digital computer for computing, without difficulty, useful approximations to the phase and unit step response, given the attenuation characteristic of the network function. The technique is applicable to those low-pass networks whose network functions have neither poles nor zeroes in the right half of the complex frequency plane, and no poles on the real frequency axis.

For a low-pass network function that satisfies the specified conditions, the application of Hilbert transforms is used in obtaining a suitable approximation of the phase of the network function. The unit step response is computed by the application of a Fourier transform which relates the real part of the network function to the response of the network function to a unit step function. A digital computer program is discussed which is used to compute the phase and unit step response from the given attenuation characteristic which is fed into the digital computer as data.

As an illustration of the technique of calculating the phase and unit step response, the phase and unit step response are computed for a specific problem and compared to the true phase and unit step response. It was found that the phase could be computed from the attenuation characteristic with an error of approximately $\mp 1.5^{\circ}$ and that the unit step response could be computed with an error of approximately one per cent.

Eight cases of variation in the attenuation characteristic of low-pass network functions are studied. In each case, families of curves of the phase and unit step response are computed corresponding to the various attenuation characteristics of the network functions.

CHAPTER I

INTRODUCTION

Often in practice the performance specifications of a network are stated in terms of the network response to a unit step function. Typical questions the network designer must answer concern the delay, rise time, amount of overshoot, and frequency and amount of decay associated with transient oscillations. Many times the designer knows only the magnitude or attenuation characteristic for the network he is designing, and a method of obtaining the phase and unit step response associated with a particular magnitude characteristic would help furnish additional needed information.

There exist empirical rules which relate characteristics of the frequency response of networks to the characteristics of their transient response (1-4)*. Furthermore, A. H. Zemanian has developed a set of theorems stating bounds existing on the time and frequency response of various types of networks (5-7). The bounds defined in Zemanian's theorems are not approximate relations but definite restrictions that responses of the appropriate networks must obey.

Some of the practical implications of Zemanian's theorems are listed below:

(1) Any RC two-terminal network cannot have a percentage overshoot or undershoot in its voltage response to a unit input of current greater

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* Numbers in parentheses refer to the bibliography.

than one hundred per cent. Furthermore, if the driving point impedance of this RC network assumes a value k as the angular frequency ω goes to infinity, then the percentage overshoot or undershoot must be less than $100(r-k)/r$, where r is the final value of the step function.

(2) If the magnitude or real part of a network function at any angular frequency, $\omega \neq 0$, is greater than its value at $\omega = 0$, then the unit step response cannot be monotonic increasing.

(3) Any network function whose real part on the positive real-frequency axis increases with frequency cannot have an overshoot in its step response greater than eighteen per cent, nor can its rise time, from the time that the input step is applied to the time that the response first crosses the final value line, be less than $1.22(r-k)c$, where r is the final value of the step response and k and $1/c$ are the constant term and the coefficient of the $1/s$ term, respectively, in the inverse power-series expansion of the network function

$$H(s) = \frac{N(s)}{D(s)} .$$

(4) Given the maximum size of any overshoot or undershoot of the step response of a network whose network function is positive real, a lower bound on the rise time from zero to the final value exists. Similarly lower bounds on the settling time exist.

It is well known that linear networks cannot have arbitrary frequency characteristics. For the case of minimum reactance or susceptance networks, many relations have been found between the real and imaginary parts of the network immittance and similar relations exist between the magnitude and phase functions in the minimum phase case.

The objective of this investigation was to study the magnitude, phase, and unit step response of low-pass networks and to illustrate a technique of general applicability using a digital computer for calculating the phase and unit step response, given the attenuation characteristic of the network function. The technique will be applicable to those low-pass networks whose network functions have neither poles nor zeroes in the right half plane, and no poles on the real frequency axis.

It is assumed that a unit step of current or voltage is impressed upon a low-pass network and a corresponding voltage or current response is taken as the output so that the network function is either an impedance or an admittance. Further, it is assumed that the input is applied at time, $t = 0$, so that the unit step response is identically equal to zero for negative values of time.

It is shown in Chapter II that the phase of a low-pass network function can be calculated from the attenuation characteristic of the network function by Hilbert transforms and that the response of a low-pass network to a unit step function can be calculated by a Fourier transform which relates the unit step response to the real part of the network function. Also discussed in Chapter II is the digital computer program used to calculate the phase and unit step response of a low-pass network function.

An illustration of the method of calculating the phase and unit step response is given in Chapter III, where the calculated and true phase and unit step response are compared for a specific problem. Eight cases of variation in the attenuation characteristic of low-pass network functions are studied. Families of curves of the attenuation, phase and unit step response for these eight cases are presented in Chapter IV and discussed in Chapter V.

CHAPTER II

PROCEDURE

Calculation of phase characteristic.--The magnitude of the network immittance function is specified by a plot of the attenuation characteristic, in decibels, versus the logarithm of frequency. From the attenuation characteristic, the associated minimum phase characteristic is calculated by an application of Hilbert transforms (Appendix I), described by Bode (8).

The attenuation characteristic, in decibels, plotted versus the logarithm of frequency is approximated by a series of straight lines. The phase is then determined by summing up the individual contributions of each elementary straight line segment to the total phase.

The most elementary straight-line characteristic which can be used to construct a given straight-line approximation is that in which the attenuation plotted against the logarithm of frequency is constant on one side of a prescribed frequency, ω_0, and has a constant slope thereafter. Such a characteristic is called by Bode a "semi-infinite constant slope" characteristic. A semi-infinite unit slope of attenuation is one in which the attenuation changes 6 decibels per octave, or 20 decibels per decade. The phase in radians at frequency ω_c, associated with a semi-infinite unit slope of attenuation commencing at ω_0, is given by (See Appendix I)

$$B(x_c) = \frac{2}{\pi}\left(x_c + \frac{x_c^3}{9} + \frac{x_c^5}{25} + \ldots\right) \qquad (1)$$

where

$$x_c = \frac{\omega_c}{\omega_0}, \; x_c < 1.$$

If $x_c = 1$, then $B(x_c) = \pi/4$.

If $\omega_c > \omega_0$, then $B(x_c)$ is given by

$$B(x_c) = \frac{\pi}{2} - B(x_c') \tag{2}$$

where

$$x_c' = \frac{\omega_0}{\omega_c}, \; x_c' < 1.$$

As an approximation to Equation 1 we have (See Appendix I)

$$B(x_c) \doteq \frac{2}{\pi}\left(x_c + \frac{x_c^3}{9} + \frac{x_c^5}{25} + \frac{x_c^7}{49} + \frac{x_c^9}{81}\right), \; 0 \le x_c \le 0.414 \tag{3}$$

$$B(x_c) \doteq \frac{\pi}{4} - \frac{1}{\pi}\log_e(x_c)\log_e\left(\frac{1-x_c}{1+x_c}\right) - \frac{2}{\pi}\left[\left(\frac{1-x_c}{1+x_c}\right) + \frac{1}{9}\left(\frac{1-x_c}{1+x_c}\right)^3 \right. \tag{4}$$

$$\left. + \frac{1}{25}\left(\frac{1-x_c}{1+x_c}\right)^5 + \frac{1}{49}\left(\frac{1-x_c}{1+x_c}\right)^7 + \frac{1}{81}\left(\frac{1-x_c}{1+x_c}\right)^9\right], \; 0.414 \le x_c < 1.$$

The phase associated with a semi-infinite slope of attenuation, of slope k, is k times the phase of a semi-infinite unit slope of attenuation given by Equation 3 or 4.

The total phase $\theta(\omega)$ at a particular frequency ω is then given by the sum of the phase at frequency ω associated with each of the semi-infinite constant slopes of attenuation which together make up the straight-line approximation. Thus, for the general straight-line approximation having slopes

$$k_1, k_2, \ldots, k_n,$$

$$\theta(\omega) = k_1(\theta_0-\theta_1) + k_2(\theta_1-\theta_2) + \ldots + k_n(\theta_{n-1}-\theta_n), \tag{5}$$

where

θ_n is the phase at frequency ω_n associated with the semi-infinite slope of attenuation commencing at frequency ω_n and extending to $\omega = \infty$ and is calculated using Equation 3 or 4,

and

k_n is the slope of the straight-line approximation between ω_{n-1} and ω_n given by

$$k_n = \frac{A_n - A_{n-1}}{20 \log_{10}(\omega_n/\omega_{n-1})}, \tag{6}$$

where

A_n is the attenuation at frequency ω_n on the straight-line approximation.

<u>Calculation of unit step response</u>.--Having determined the phase associated

with the given attenuation characteristic, the value of the real part of the immittance function, $R(\omega)$, at a particular frequency ω is given by

$$R(\omega) = 10^{(A(\omega)/20)} \cos \theta(\omega), \tag{7}$$

where

$A(\omega)$ is the attenuation of the immittance function at a particular frequency ω obtained from the attenuation characteristic plotted versus the logarithm of frequency,

and

$\theta(\omega)$ is the phase of the immittance function at the particular frequency ω given by Equation 3 or 4.

The unit step response, $A(t)$, is then given by the Fourier transform (See Appendix II)

$$A(t) = \frac{2}{\pi} \int_0^{\infty} \frac{R(\omega)\sin\omega t}{\omega} \, d\omega, \quad t \geq 0, \tag{8}$$

which relates the real part of the immittance function, $R(\omega)$, along the real frequency axis to the unit step response, $A(t)$. The value of this integral is obtained by using Simpson's Rule of numerical integration (9).

If $y_0, y_1, y_2, \ldots,$ are values of $y = f(x)$ at equally-spaced points $x_0, x_1, x_2, \ldots,$ with interval h, then Simpson's Rule can be written as

$$\int_{x_0}^{x_{2m}} y \, dx = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{2m-1} + y_{2m}\right). \tag{9}$$

The main restriction on Simpson's Rule is that it must be applied on an even number of intervals. The real part of the immittance function, $R(\omega)$, is known on a logarithmic frequency scale and to meet the requirement of equally-spaced points, the integration is performed by decades of frequency, allowing the interval h to change in each decade.

It was found that an error of approximately one per cent in the unit step response resulted if the upper limit of integration was 10.0 radians per second and the attenuation was 20 db down at this frequency. It was also found that a lower limit of integration of 10^{-6} radians per second is necessary to prevent droop in the unit step response. The response labeled (c) in Figure 1 resulted from integration with a lower limit of 0.01 radians per second.

The attenuation characteristic is plotted over the normalized frequency range 0.01 to 10.0 radians per second on three cycle semi-log graph paper. The attenuation is read from this plot at 19 equally-spaced frequencies in the decade from 0.01 to 0.1, 37 equally-spaced frequencies in the decade from 0.1 to 1.0 and 37 equally-spaced frequencies in the decade 1.0 to 10.0. The attenuation is assumed constant and equal to the attenuation at zero frequency for 12 equally-spaced frequencies in the four decades below 0.01. If only 37 points are used for the integration over the decade from 1.0 to 10.0 radians per second, the resulting unit step response is found to have excessive error. This is shown by the response labeled (a) in Figure 1. Using linear interpolation to calculate the attenuation at 181 equally-spaced frequencies from the 37 values of attenuation read from the plotted attenuation characteristic and integrating over 181 points greatly reduces this error. The responses of Figure 1

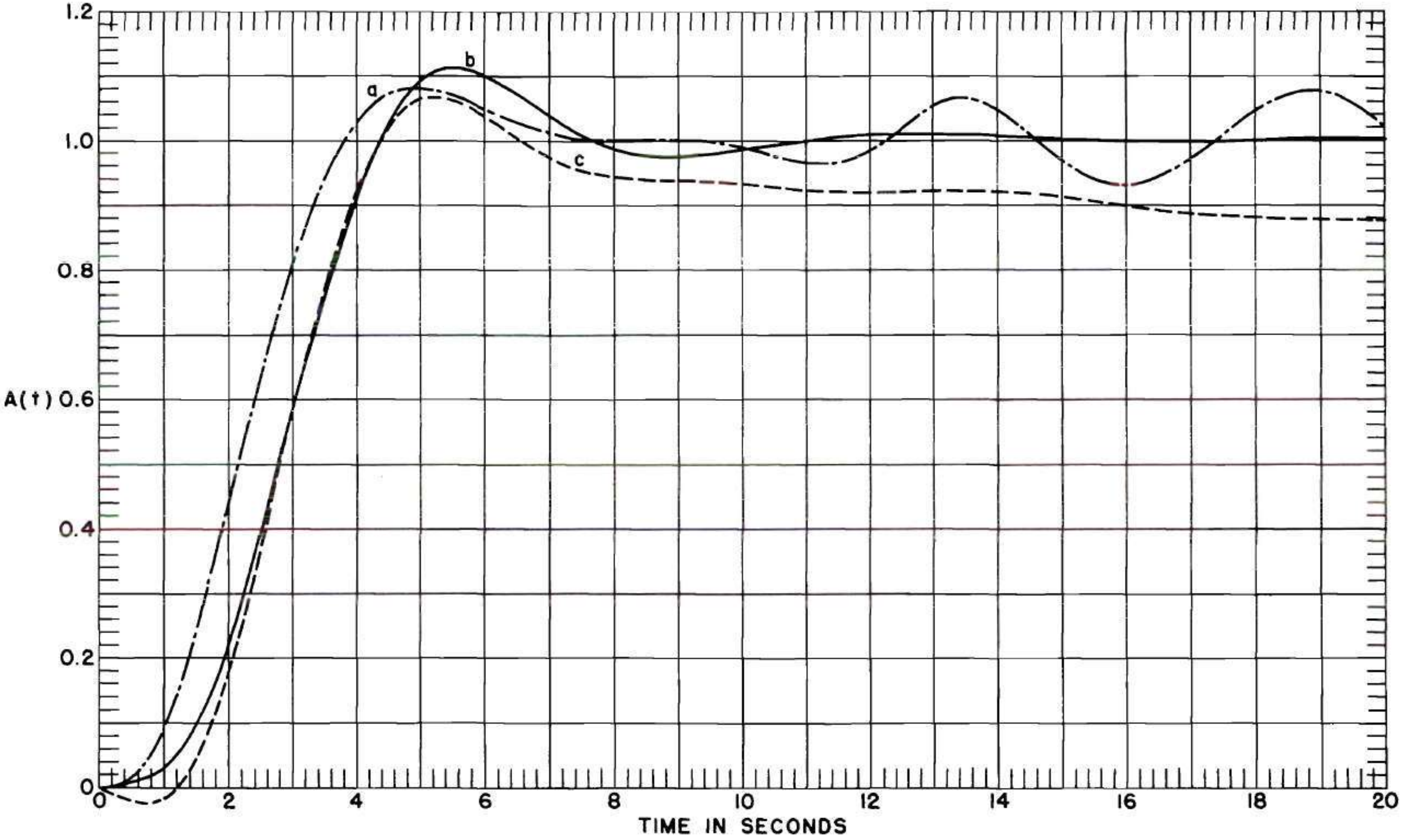

Figure 1. Comparison of Step Responses

were calculated for a third order Butterworth function and the response labeled (b) departs from the true response by approximately one per cent.

Digital computer program.--The above described method of calculating the phase and unit step response of a low-pass network function is carried out in a straight forward manner with the aid of the Burroughs 220 Data-Processing System. A program was written in a representation of ALGOL (Appendix III) for use with the Burroughs Algebraic Compiler (10). The program and data for a given problem are fed into the computer on punched cards and the phase and unit step response are printed out on the Line Printer.

The following data from the attenuation characteristic are punched on cards:

(1) The number of critical points on the straight-line approximation to the attenuation characteristic.

(2) The critical frequencies.

(3) The attenuation at the critical frequencies on the straight-line approximation.

(4) The attenuation at zero frequency.

(5) The attenuation at 19 equally-spaced frequencies in the decade 0.01 to 0.1, 37 equally-spaced frequencies in the decade 0.1 to 1.0 and 37 equally-spaced frequencies in the decade 1.0 to 10.0.

Total running time for the program is approximately eight minutes. Two minutes are required for compilation of the symbolic program into machine language and six minutes for calculating the phase and unit step response.

CHAPTER III

SAMPLE PROBLEM

As an illustration of the method discussed in Chapter II, consider the determination of the phase and unit step response associated with the characteristic given by $20\log_{10}|Z(j\omega)|$ shown in Figure 2. The characteristic is first approximated by a series of straight lines as shown in dotted form. The critical frequencies and the attenuation at these critical frequencies are then read from the straight-line approximation and are tabulated in Table 1.

Table 1. Critical Points for Straight-Line Approximation to Characteristic of Figure 2

n	ω_n	A_n
0	0.10	0.00
1	0.25	-0.75
2	0.70	-2.75
3	0.85	-2.00
4	1.00	0.00
5	1.25	5.25
6	1.40	5.25
7	1.75	0.00
8	2.25	-5.00
9	3.50	-10.00
10	7.00	-16.75
11'	18.00	-25.00
11	∞	

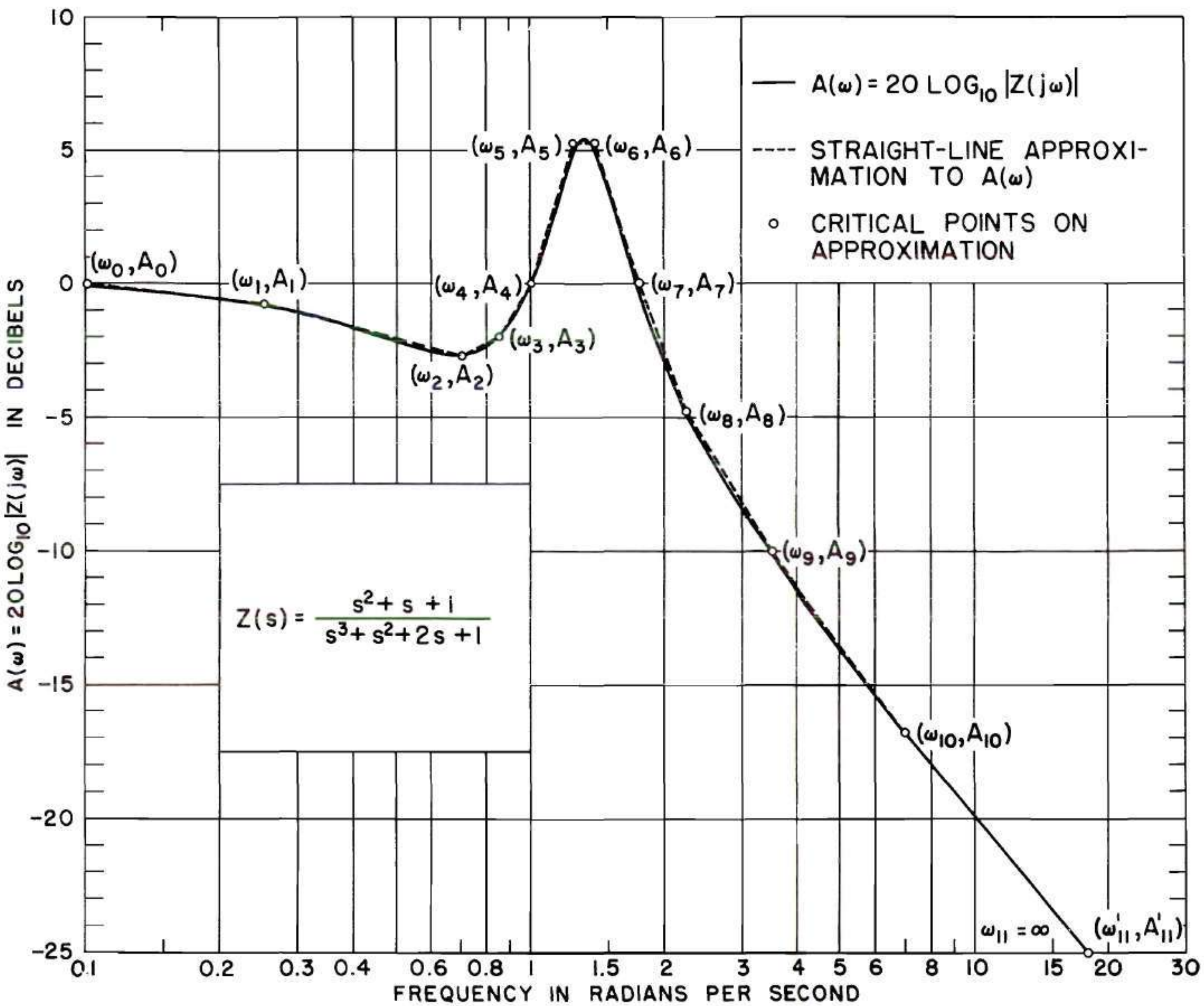

Figure 2. Attenuation Characteristic and Straight-Line Approximation

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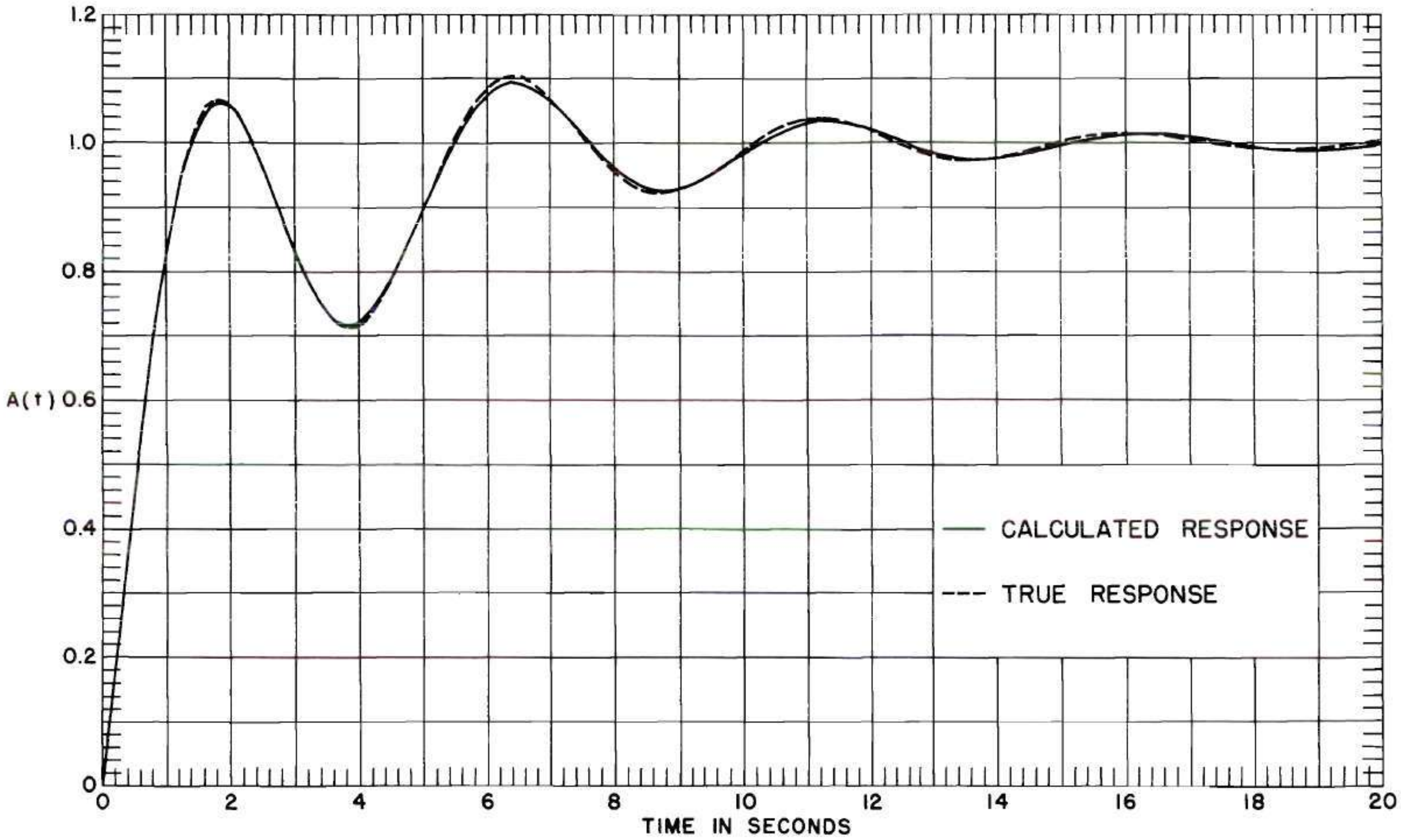

Figure 4. Comparison of Calculated and True Unit Step Response

In many problems the attenuation will have a constant slope extending to infinity. In this case, the top critical frequency will be infinity and the slope must be determined over a finite portion of this infinite slope. Note in Figure 2 that this is the case and the final slope must be determined by the change in attenuation between the critical frequencies ω_{10} and ω_{11}'. Similarly, when there is a constant slope from zero frequency, the slope must be determined over a finite portion of this slope and zero frequency taken as the first critical frequency.

Plotted in Figure 3 is the phase determined from the straight-line approximation of the characteristic of Figure 2. The true phase is shown in dotted form. The straight-line approximation to the characteristic of Figure 2 was of the order of $\mp$ 0.25 db and the maximum departure of the calculated phase from the true phase is approximately $\mp 1.5^{\circ}$. A much simpler approximation than that of Figure 2 could be used without a great loss of accuracy.

Plotted in Figure 4 is the unit step response associated with the characteristic of Figure 2. The true response is shown in dotted form. The departure of the calculated response from the true response is approximately one per cent.

CHAPTER IV

RESULTS

The attenuation, phase and unit step response for the eight cases of variation in the attenuation characteristic are shown in Figures 5 to 19. The curves are normalized with respect to both frequency and time.

Case I consists of the first six Butterworth functions having unit angular cutoff frequencies. In Case II, the network functions have flat pass bands of unit angular frequency with constant slopes of attenuation thereafter. In Case III, the network functions have flat pass bands of unit angular frequency with constant slopes of attenuation to flat stop bands 20 db, 40 db, and 60 db down. In Case IV, the network functions have flat pass bands of unit angular frequency with constant slopes of attenuation to flat stop bands starting at angular frequencies of two, four, and six radians per second. In Case V, the network functions have flat pass bands of unit angular frequency and extremely sharp cutoffs to flat stop bands 60 db down.

The network functions of Cases VI and VII have, respectively, equal-ripple pass bands and sloping pass bands. The network functions in Case VIII have a sharp null in their pass bands.

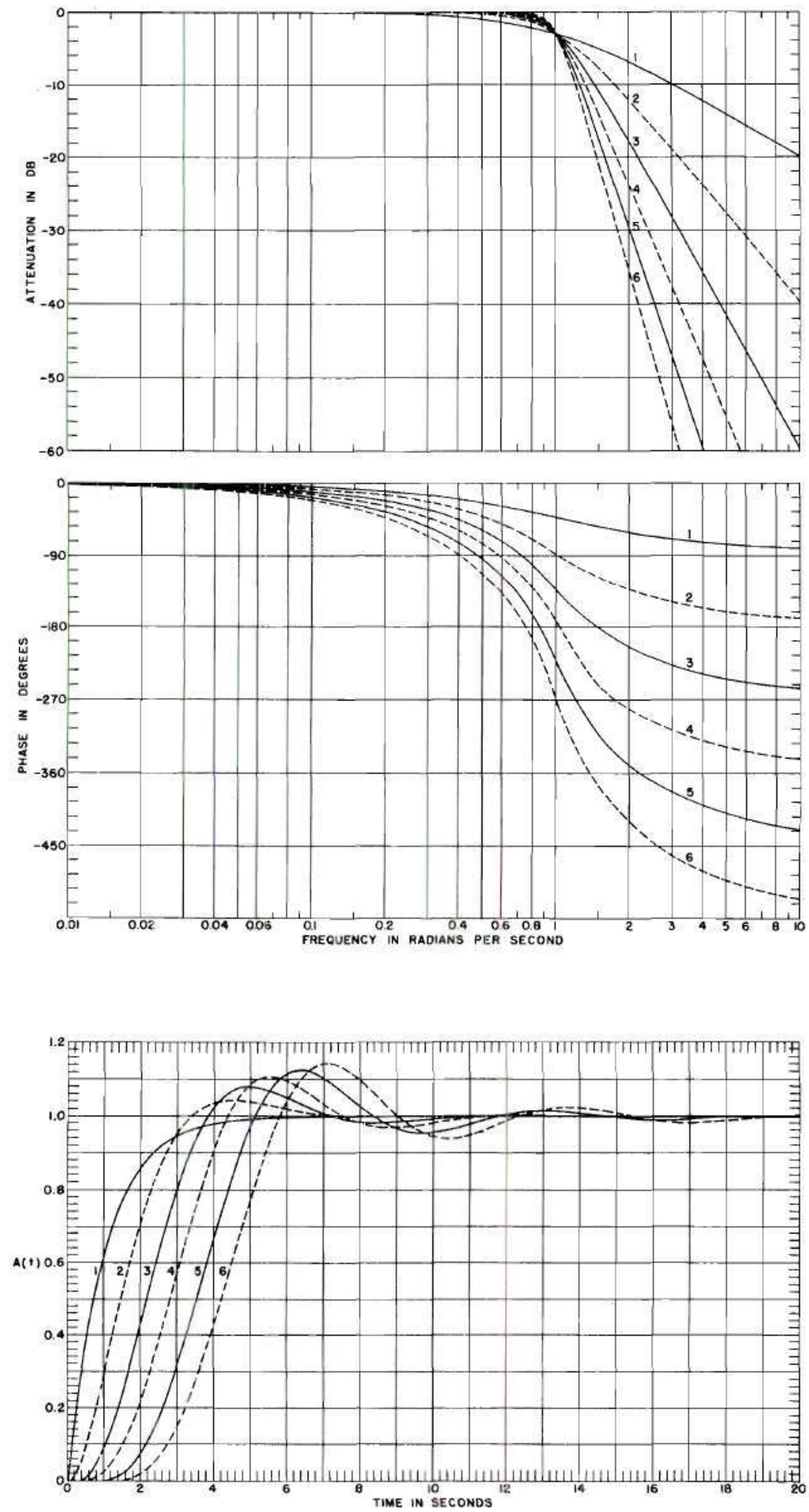

Figure 5. Attenuation, Phase and Unit Step Response for Case I

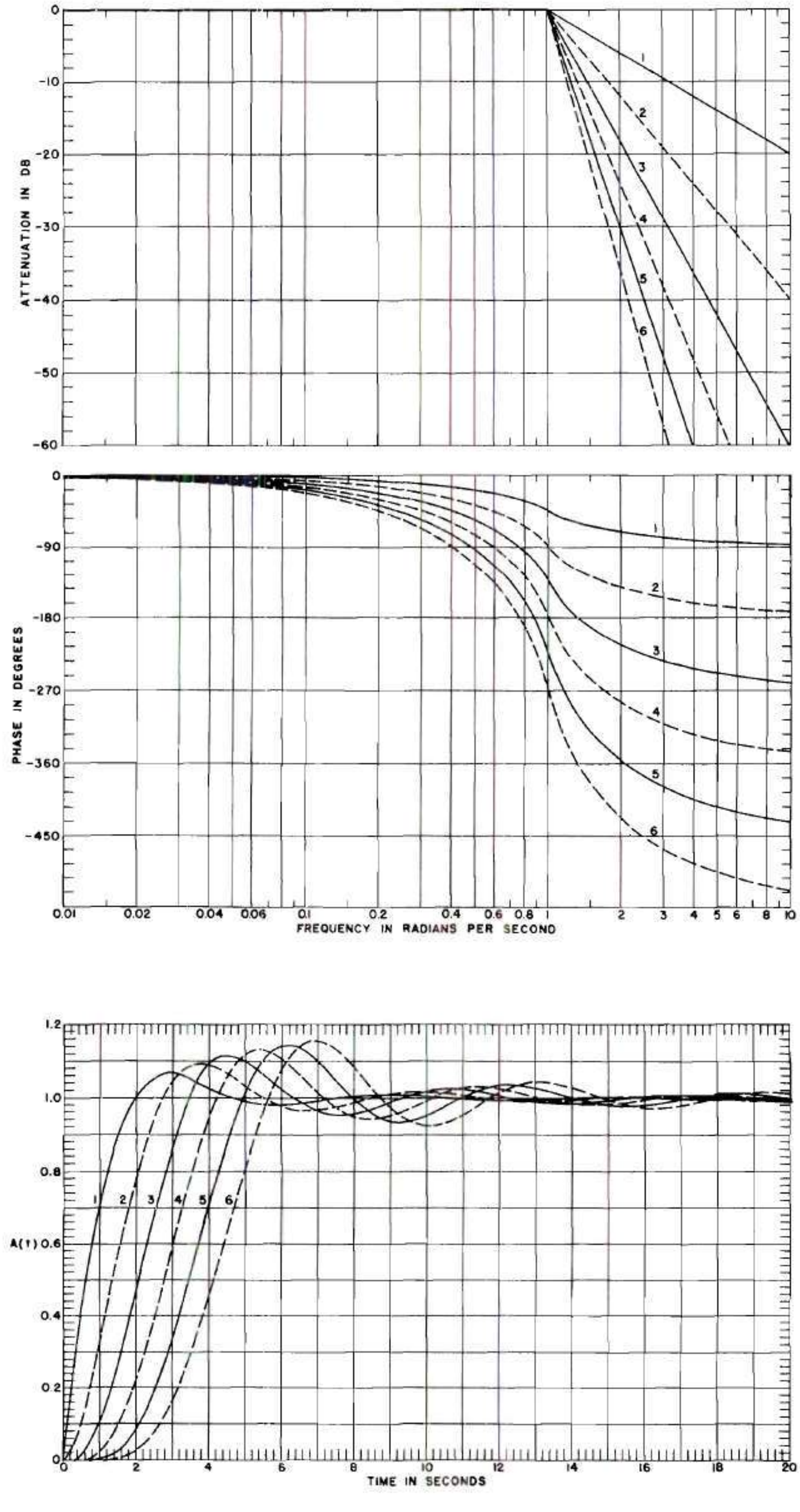

Figure 6. Attenuation, Phase and Unit Step Response for Case II

Table 2. Comparison of Step Response Characteristics of Case I and Case II

	Response No.	Percentage Overshoot	Rise Time	Delay Time	Settling Time
Case I	1	0.0	2.20	0.70	3.00
	2	4.5	2.15	1.43	2.93
	3	8.1	2.30	2.14	6.00
	4	10.9	2.46	2.78	6.24
	5	12.6	2.58	3.50	7.67
	6	14.4	2.68	4.17	10.92
Case II	1	7.0	1.39	0.51	2.50
	2	9.3	1.84	1.35	4.74
	3	11.6	2.10	2.08	5.64
	4	13.2	2.30	2.78	8.87
	5	14.4	2.47	3.45	9.90
	6	15.5	2.62	4.13	10.95

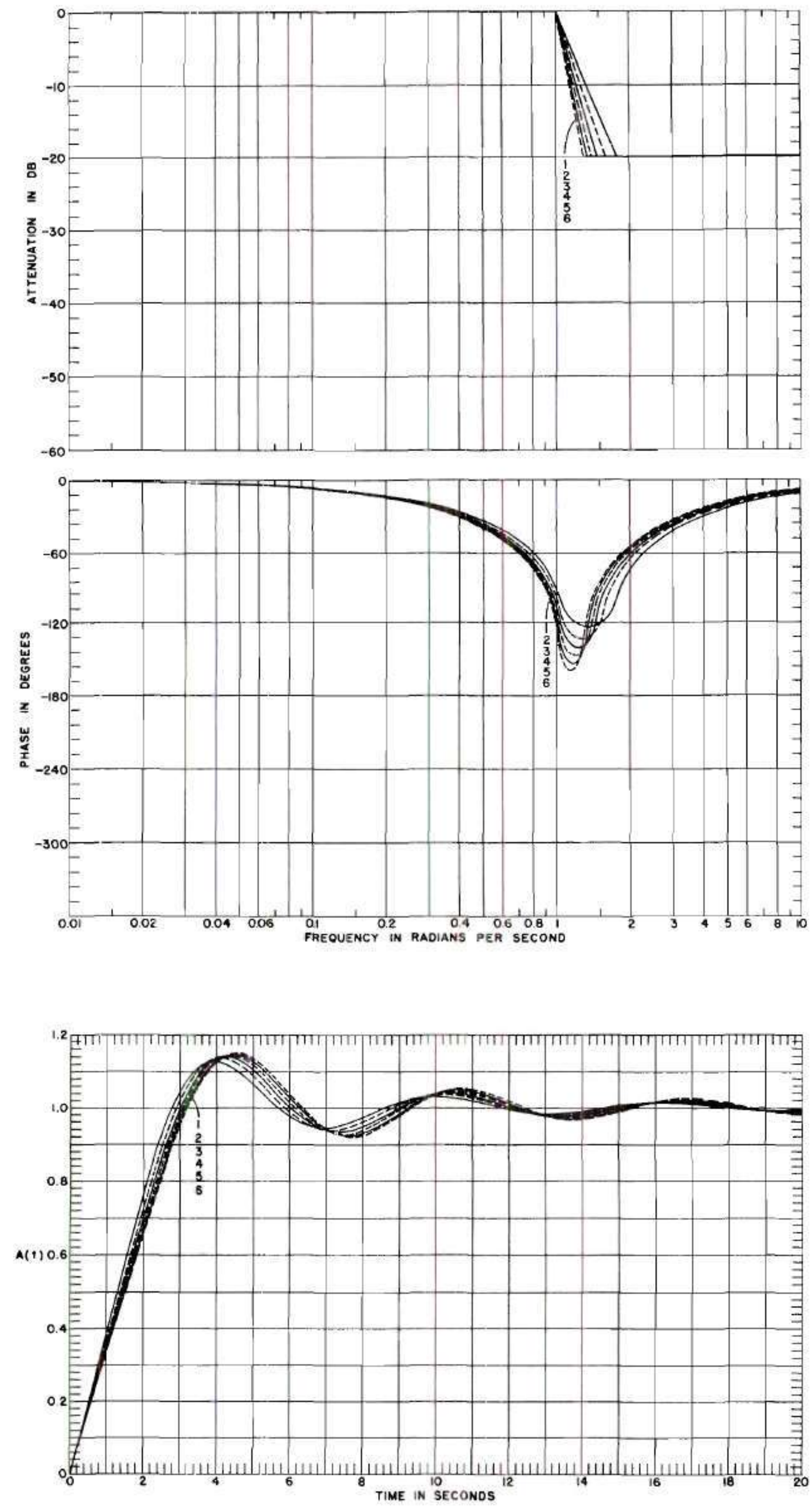

Figure 7. Attenuation, Phase and Unit Step Response for Case III(a)

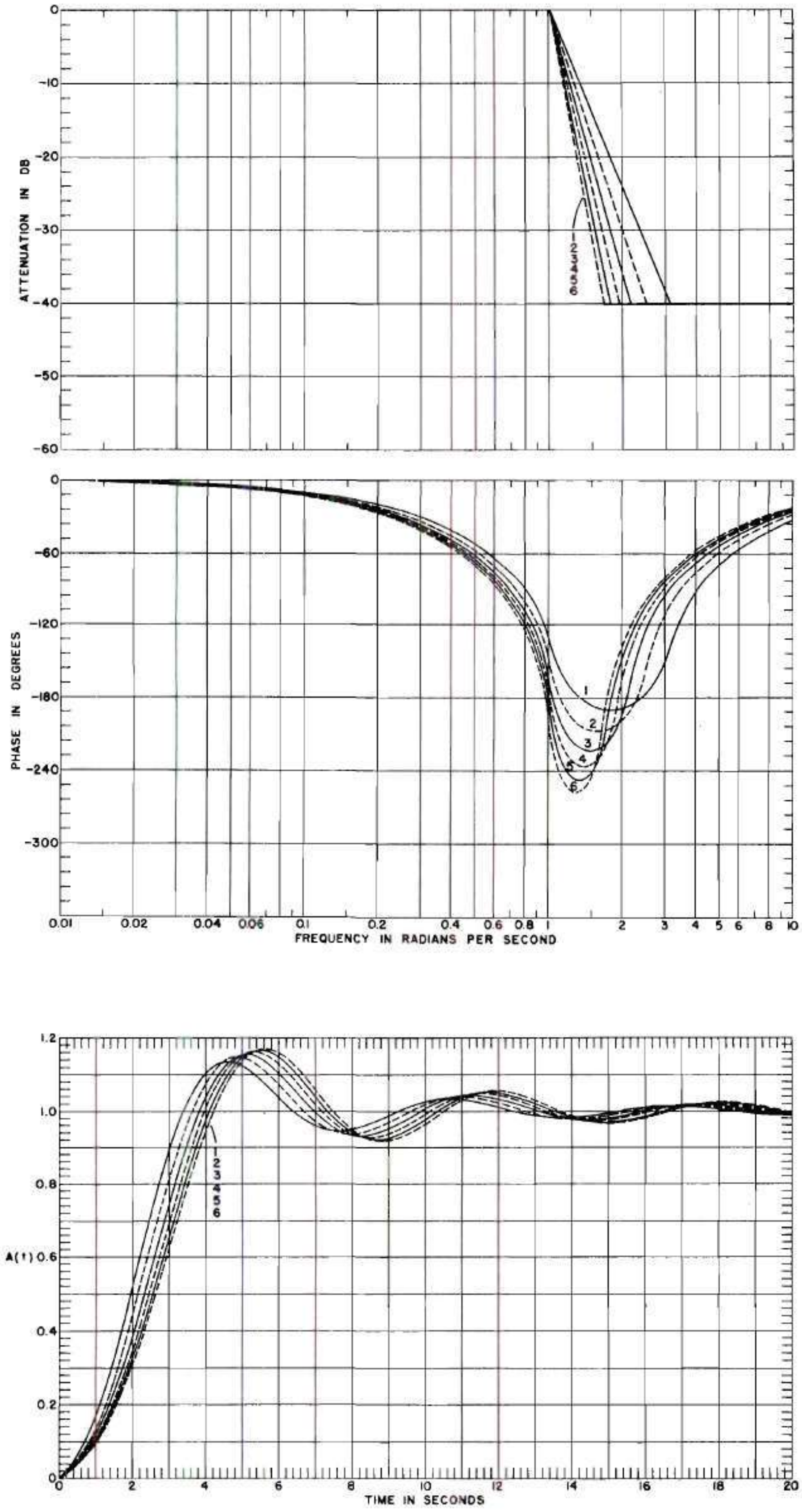

Figure 8. Attenuation, Phase and Unit Step Response for Case III(b)

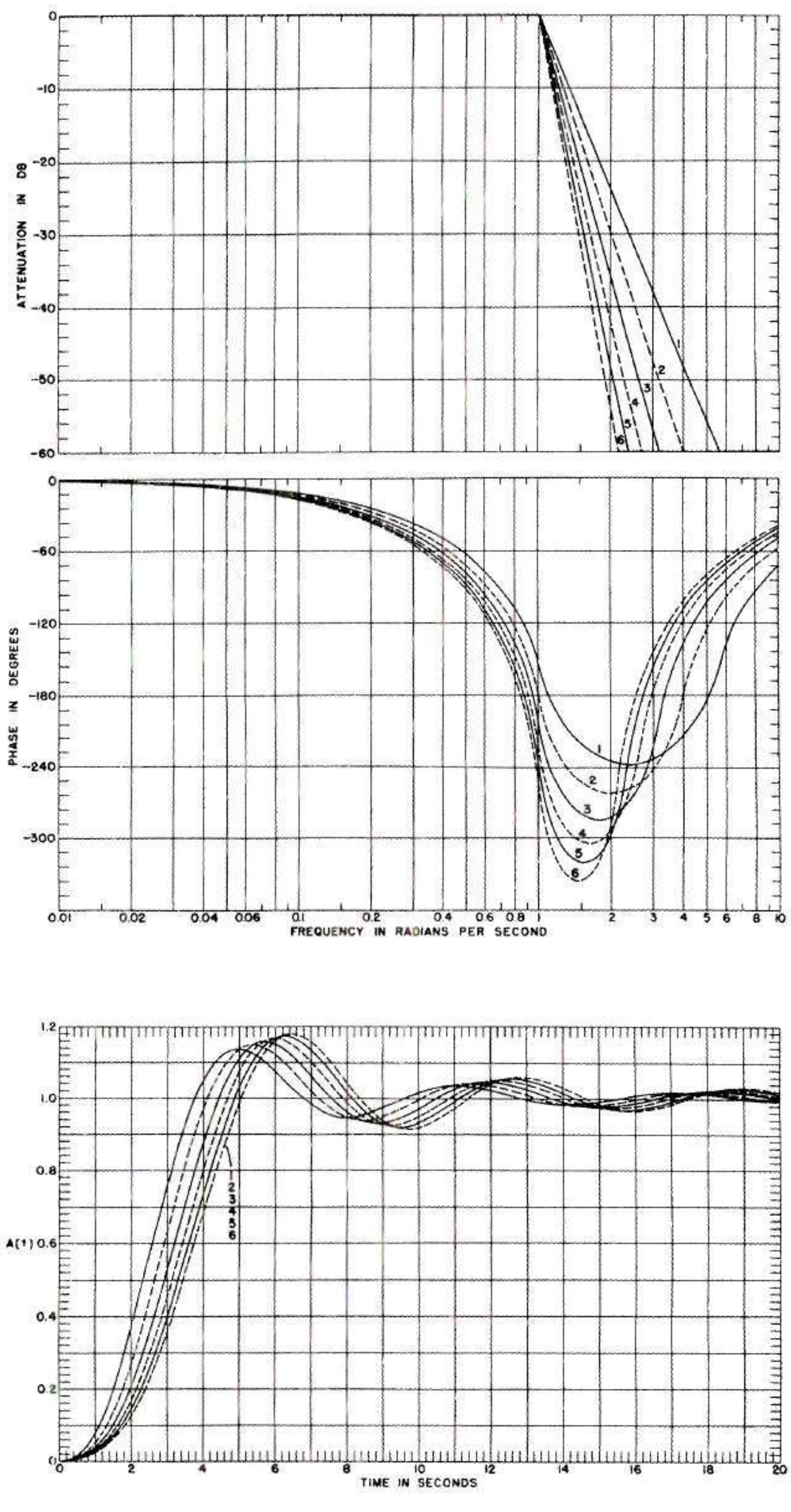

Figure 9. Attenuation, Phase and Unit Step Response for Case III(c)

Table 3. Comparison of Step Response Characteristics of Case III

	Response No.	Percentage Overshoot	Rise Time	Delay Time	Settling Time
Case III(a)	1	12.8	2.14	1.29	7.34
	2	13.8	2.25	1.38	7.74
	3	14.1	2.34	1.43	8.05
	4	14.3	2.41	1.47	8.27
	5	14.6	2.47	1.50	8.42
	6	14.9	2.52	1.53	10.95
Case III(b)	1	13.0	2.21	1.96	8.08
	2	14.3	2.40	2.15	8.62
	3	15.2	2.58	2.23	9.02
	4	15.7	2.65	2.46	9.33
	5	16.2	2.76	2.56	11.80
	6	16.7	2.82	2.63	12.24
Case III(c)	1	13.2	2.27	2.31	8.40
	2	14.4	2.45	2.64	9.06
	3	15.3	2.59	2.91	9.62
	4	16.2	2.72	3.12	10.07
	5	16.9	2.81	3.29	12.70
	6	17.3	2.87	3.43	13.20

Table 4. Comparison of Step Response Characteristics of Case II and Case III

	Response No.	Percentage Overshoot	Rise Time	Delay Time	Settling Time
Case II	4	13.2	2.30	2.78	8.87
	5	14.4	2.47	3.45	9.90
	6	15.5	2.62	4.13	10.90
Case III(a)	1	12.8	2.14	1.29	7.34
	2	13.8	2.25	1.38	7.74
	3	14.1	2.34	1.43	8.05
Case III(b)	1	13.0	2.21	1.96	8.08
	2	14.3	2.40	2.15	8.62
	3	15.2	2.58	2.23	9.02
Case III(c)	1	13.2	2.27	2.31	8.40
	2	14.4	2.45	2.64	9.06
	3	15.3	2.59	2.91	9.62

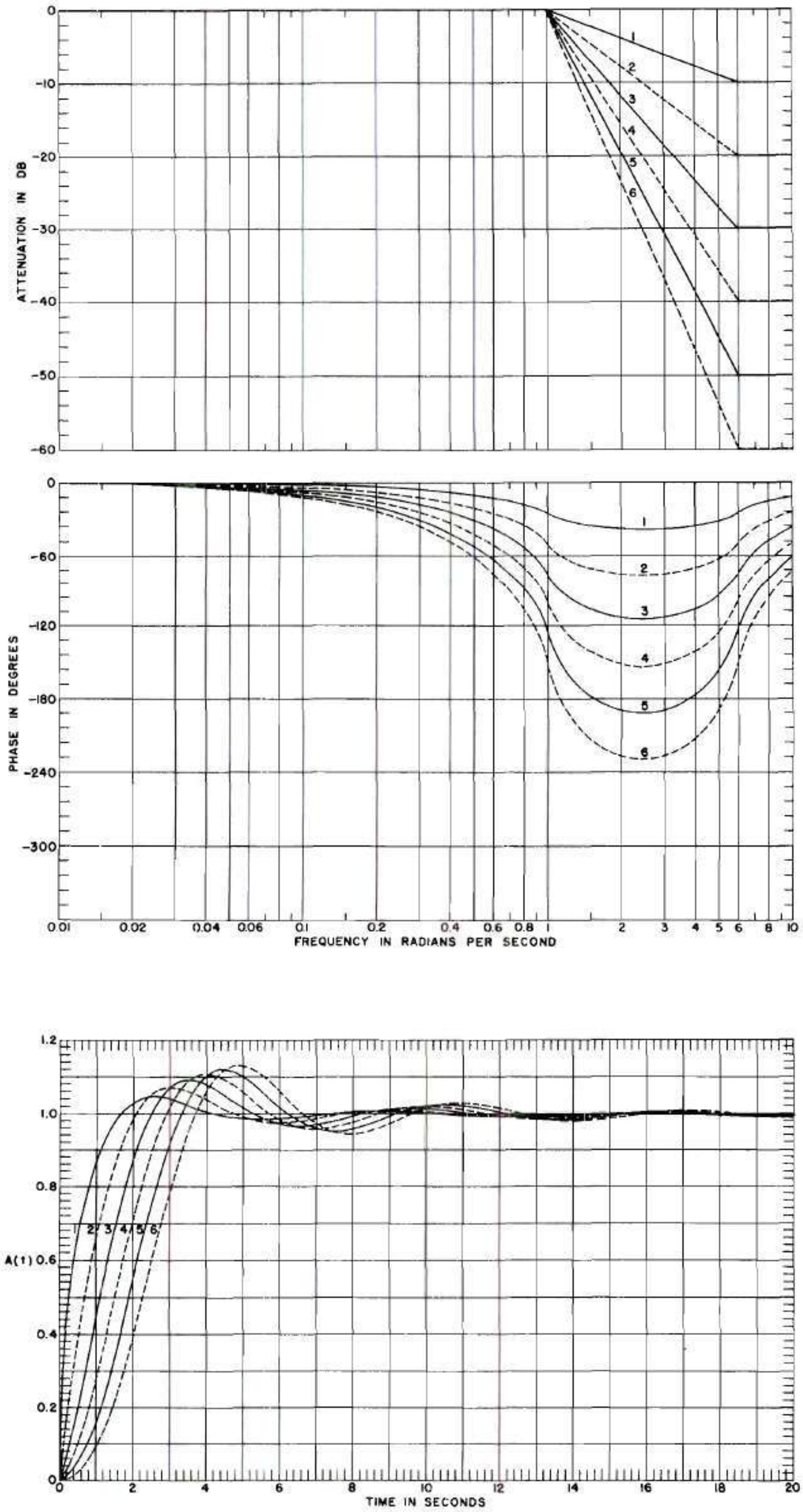

Figure 10. Attenuation, Phase and Unit Step Response for Case IV(a)

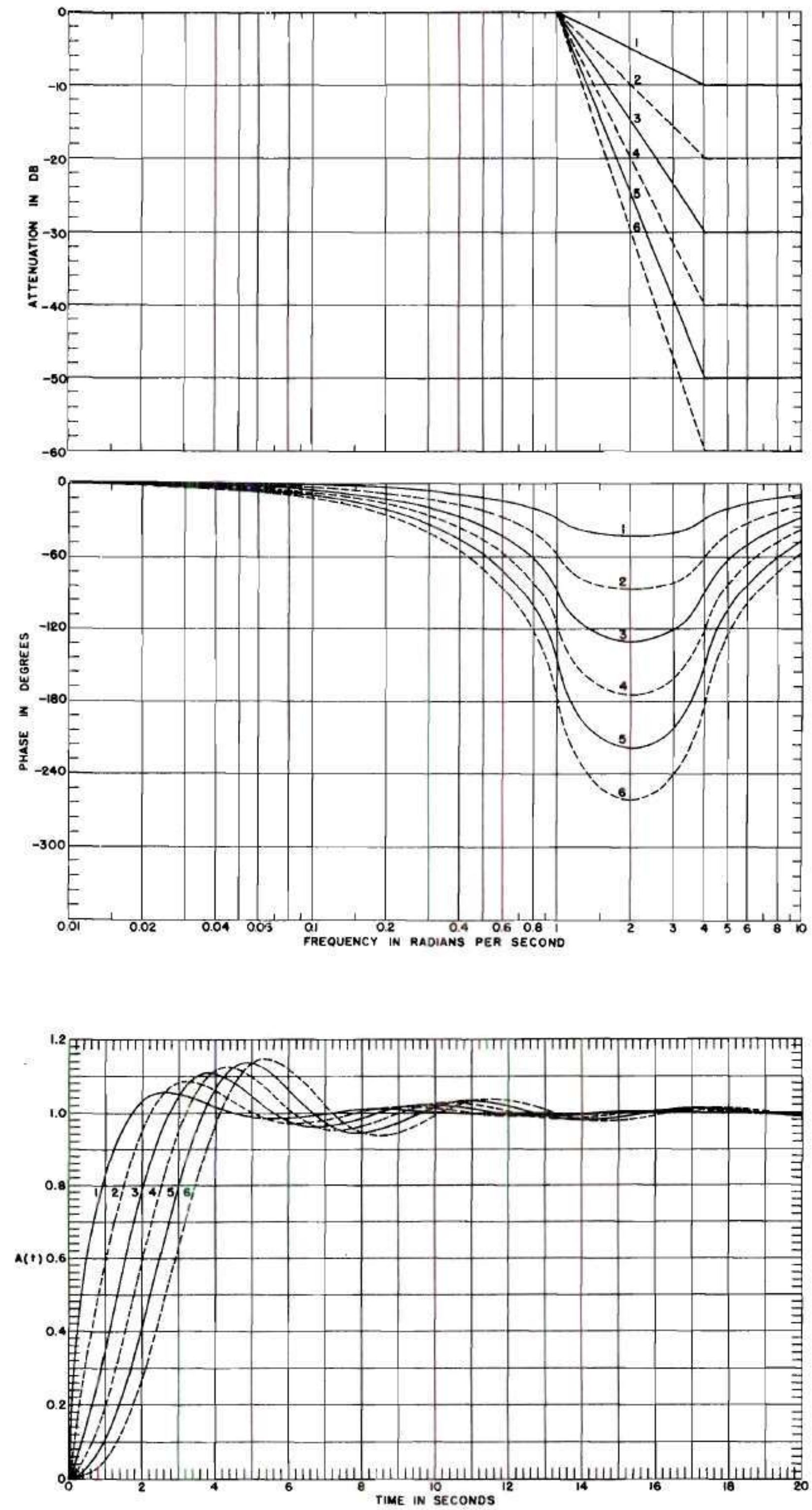

Figure 11. Attenuation, Phase and Unit Step Response for Case IV(b)

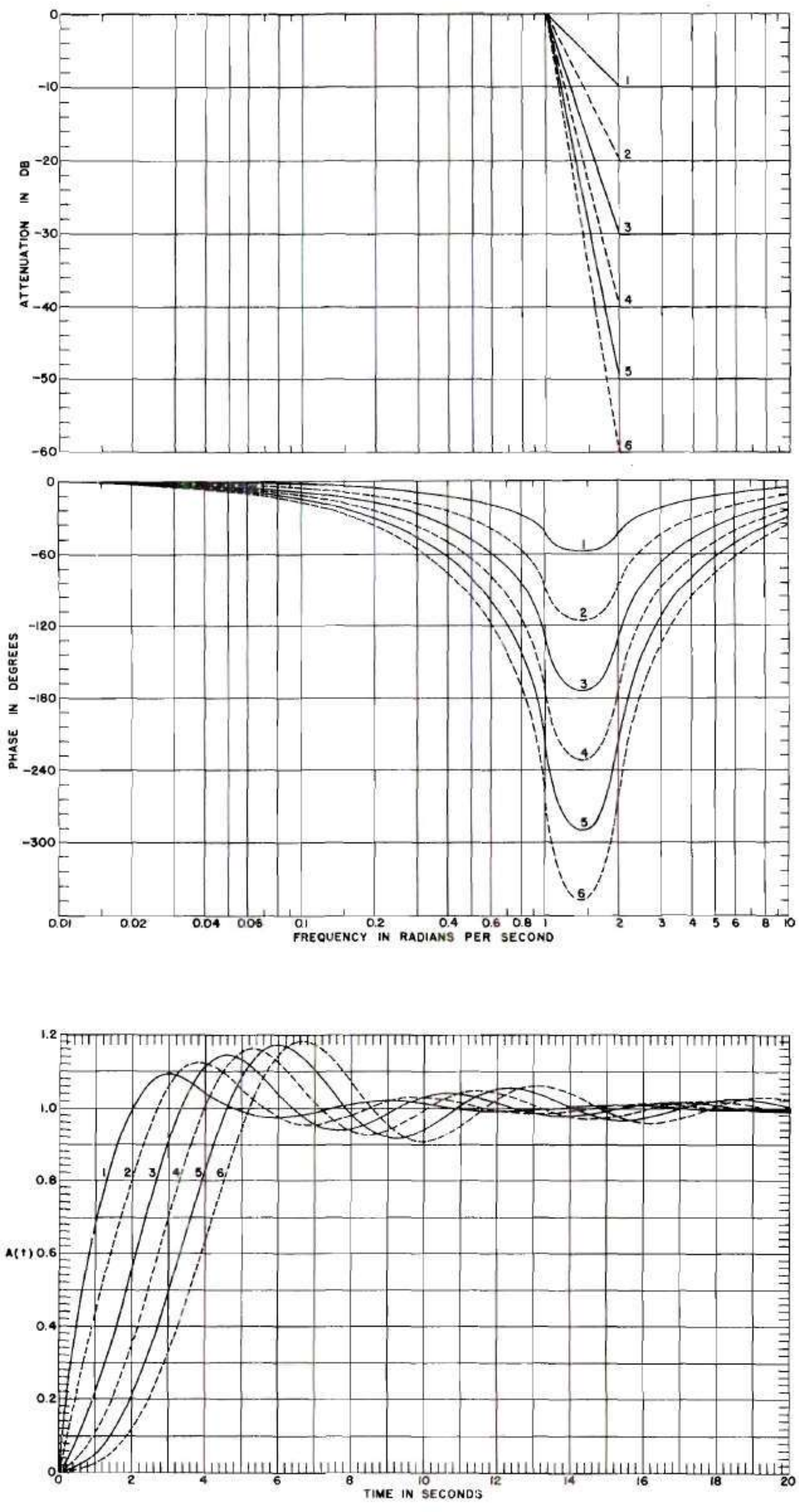

Figure 12. Attenuation, Phase and Unit Step Response for Case IV(c)

Table 5. Comparison of Step Response Characteristics of Case IV

	Response No.	Percentage Overshoot	Rise Time	Delay Time	Settling Time
Case IV(a)	1	4.8	1.08	0.24	1.33
	2	7.0	1.50	0.69	3.75
	3	9.0	1.80	1.09	4.45
	4	10.7	2.02	1.50	5.05
	5	12.0	2.16	1.88	5.57
	6	13.0	2.29	2.26	8.26
Case IV(b)	1	5.3	1.19	0.32	2.90
	2	8.3	1.66	0.81	4.10
	3	10.7	1.99	1.31	4.87
	4	12.2	2.19	1.78	7.25
	5	13.3	2.33	2.22	8.37
	6	14.4	2.46	2.64	9.05
Case IV(c)	1	8.9	1.50	0.62	3.75
	2	11.9	2.08	1.13	4.92
	3	14.0	2.48	1.82	8.28
	4	15.6	2.85	2.42	9.25
	5	16.7	3.21	3.00	12.50
	6	17.8	3.42	3.55	13.57

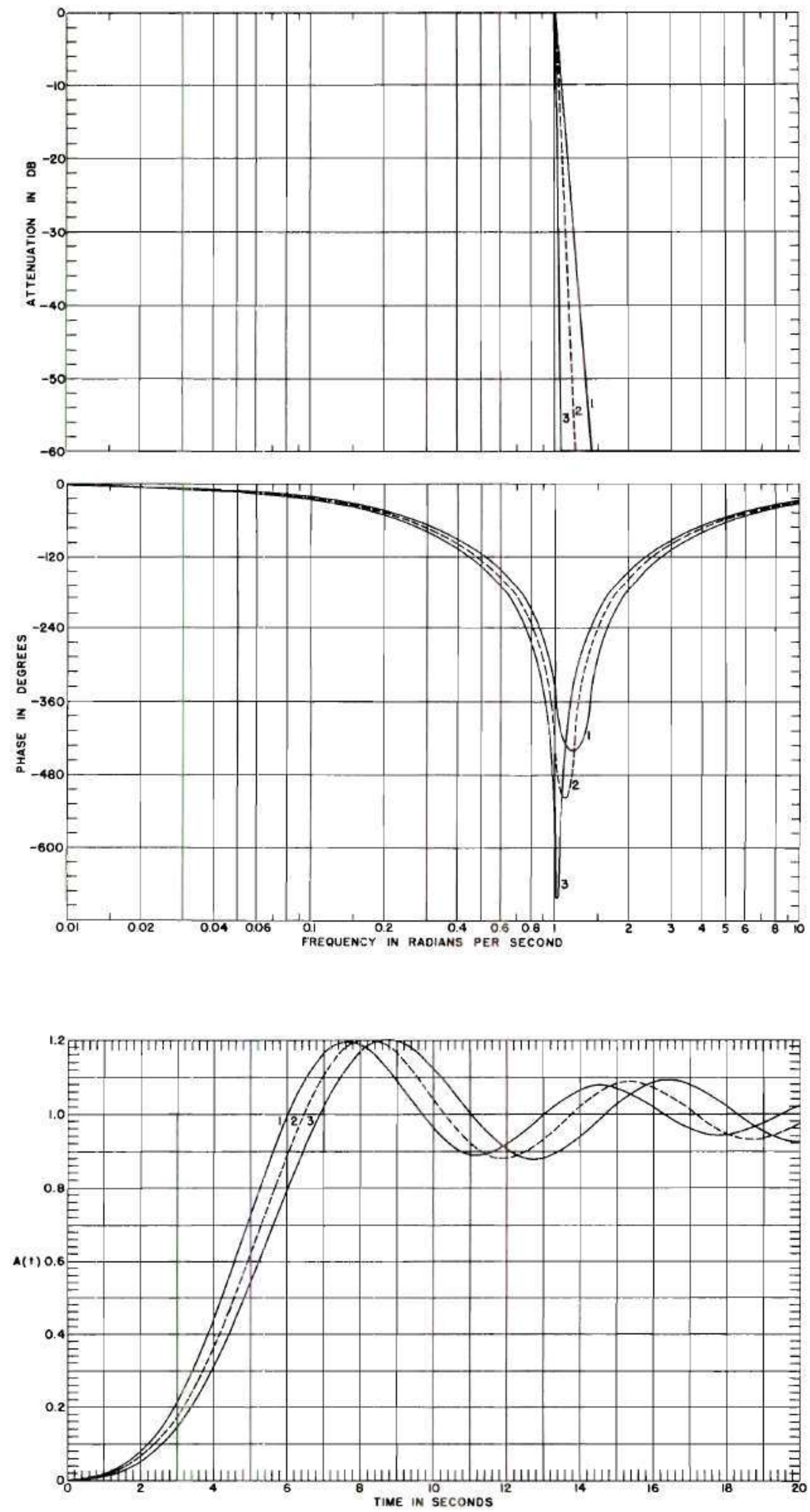

Figure 13. Attenuation, Phase and Unit Step Response for Case V

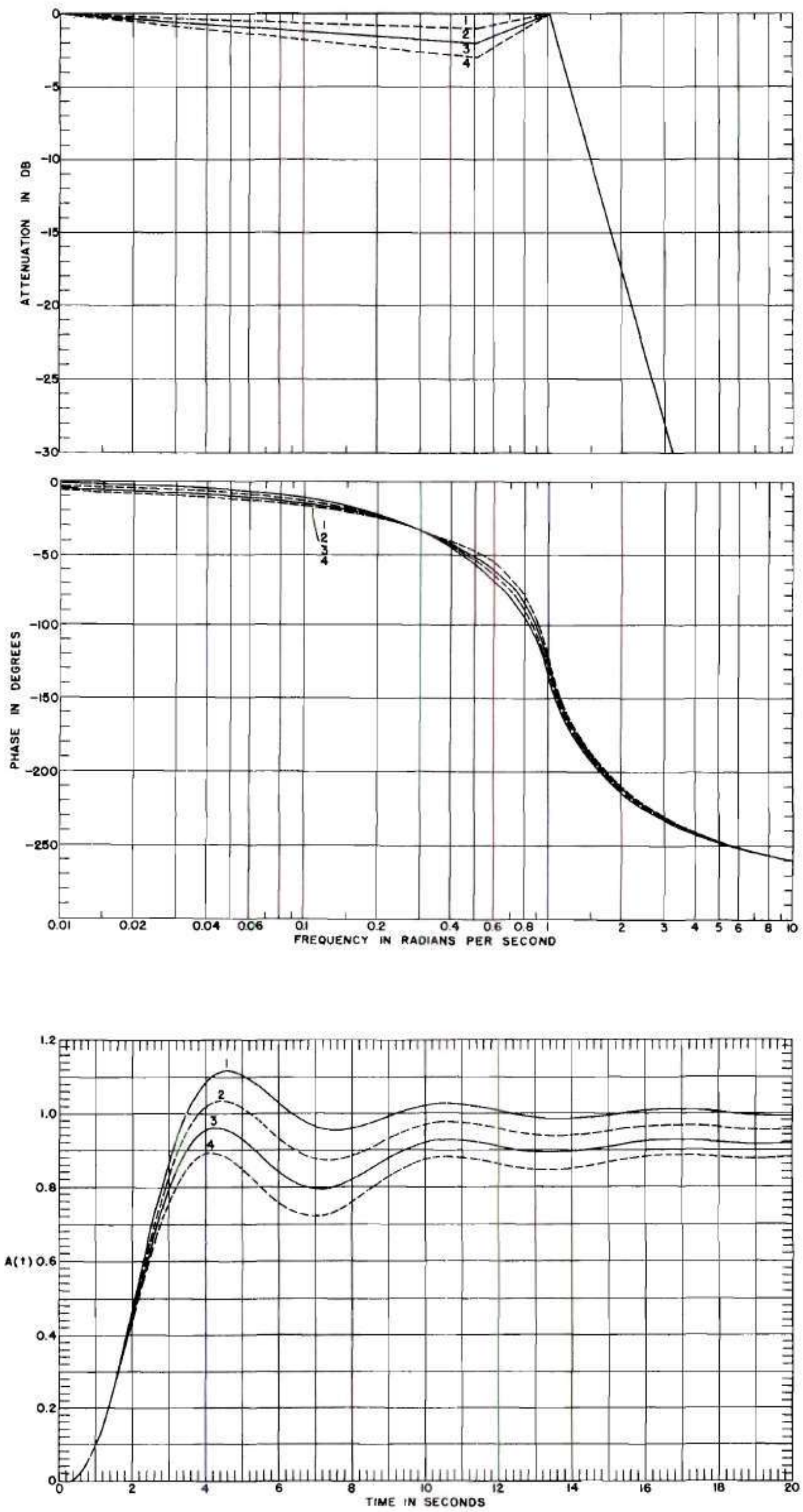

Figure 14. Attenuation, Phase and Unit Step Response for Case VI(a)

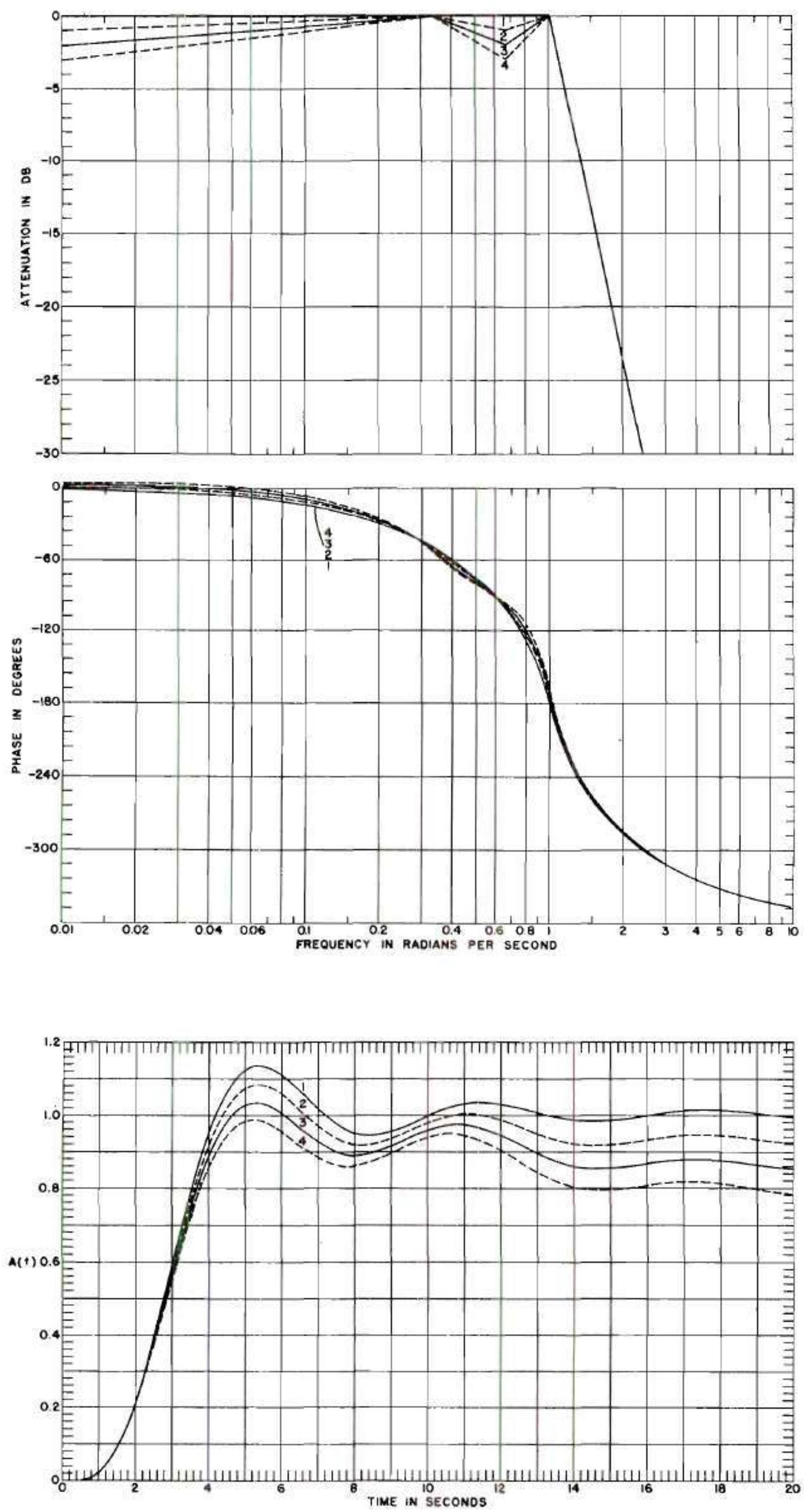

Figure 15. Attenuation, Phase and Unit Step Response for Case VI(b)

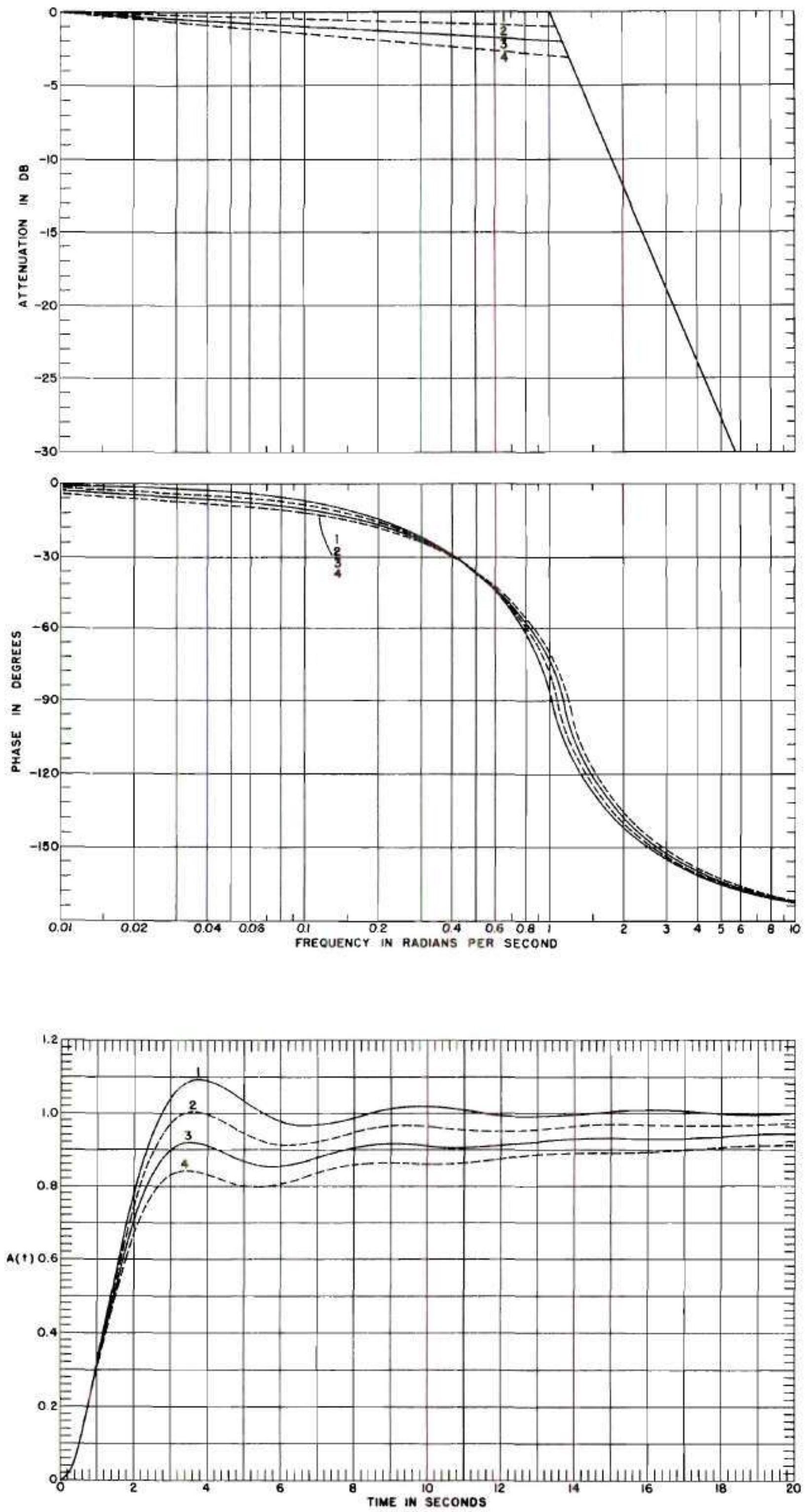

Figure 16. Attenuation, Phase and Unit Step Response for Case VII(a)

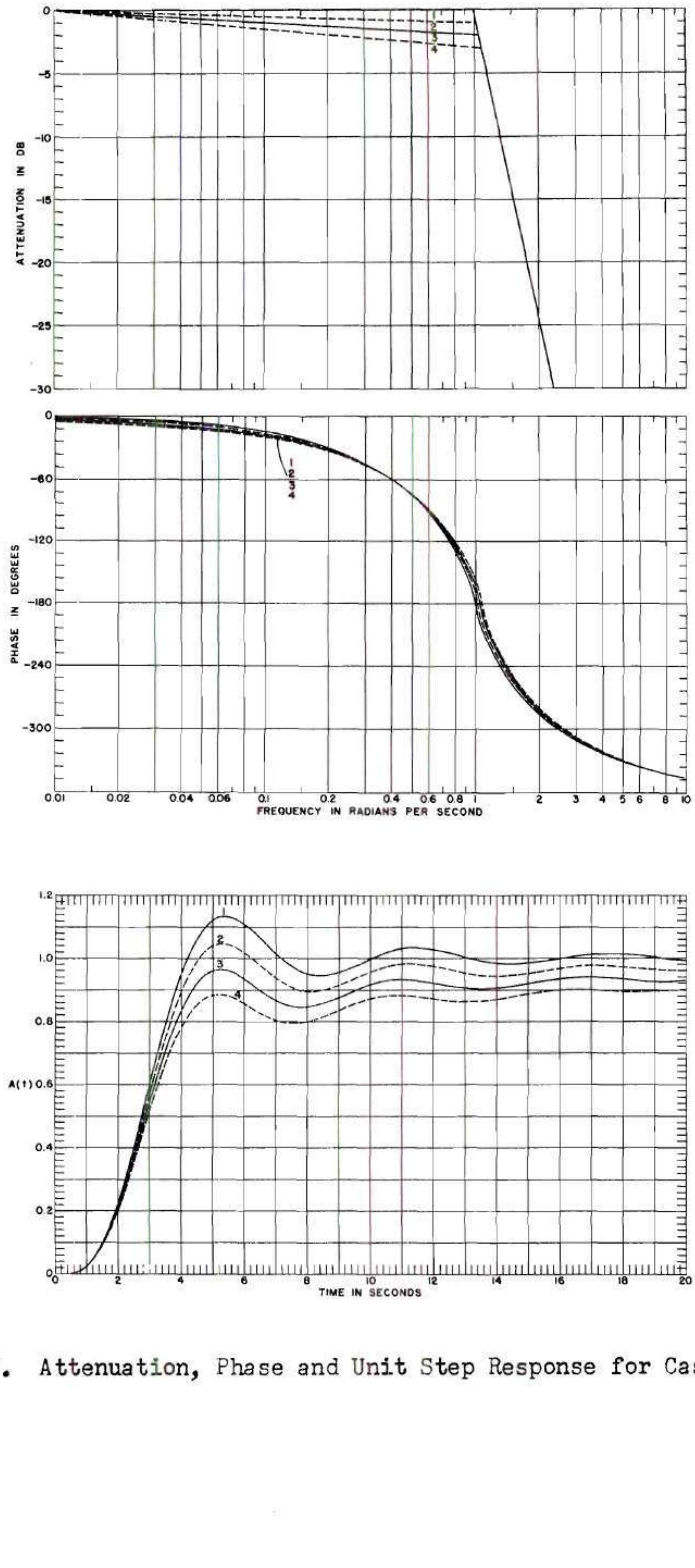

Figure 17. Attenuation, Phase and Unit Step Response for Case VII(b)

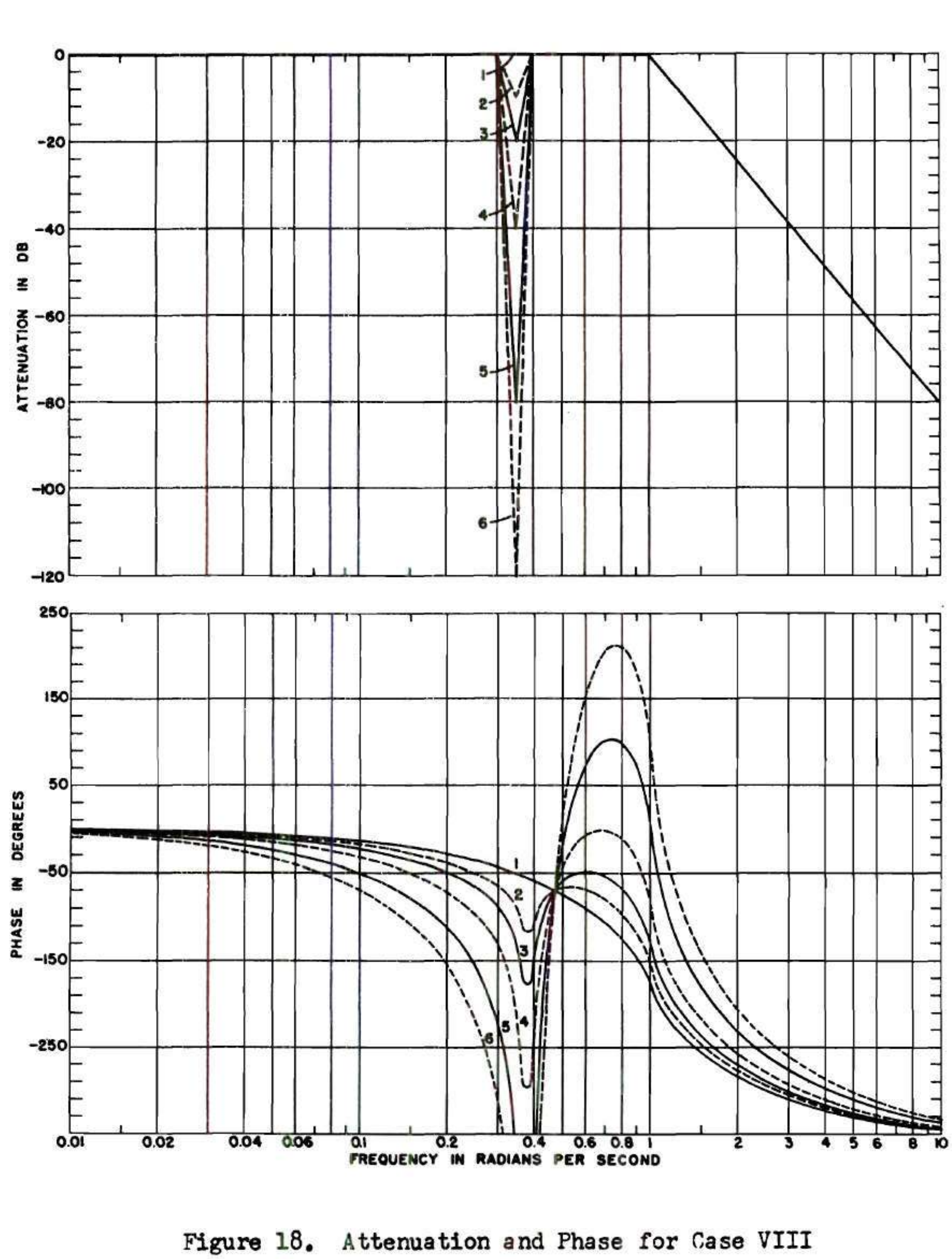

Figure 18. Attenuation and Phase for Case VIII

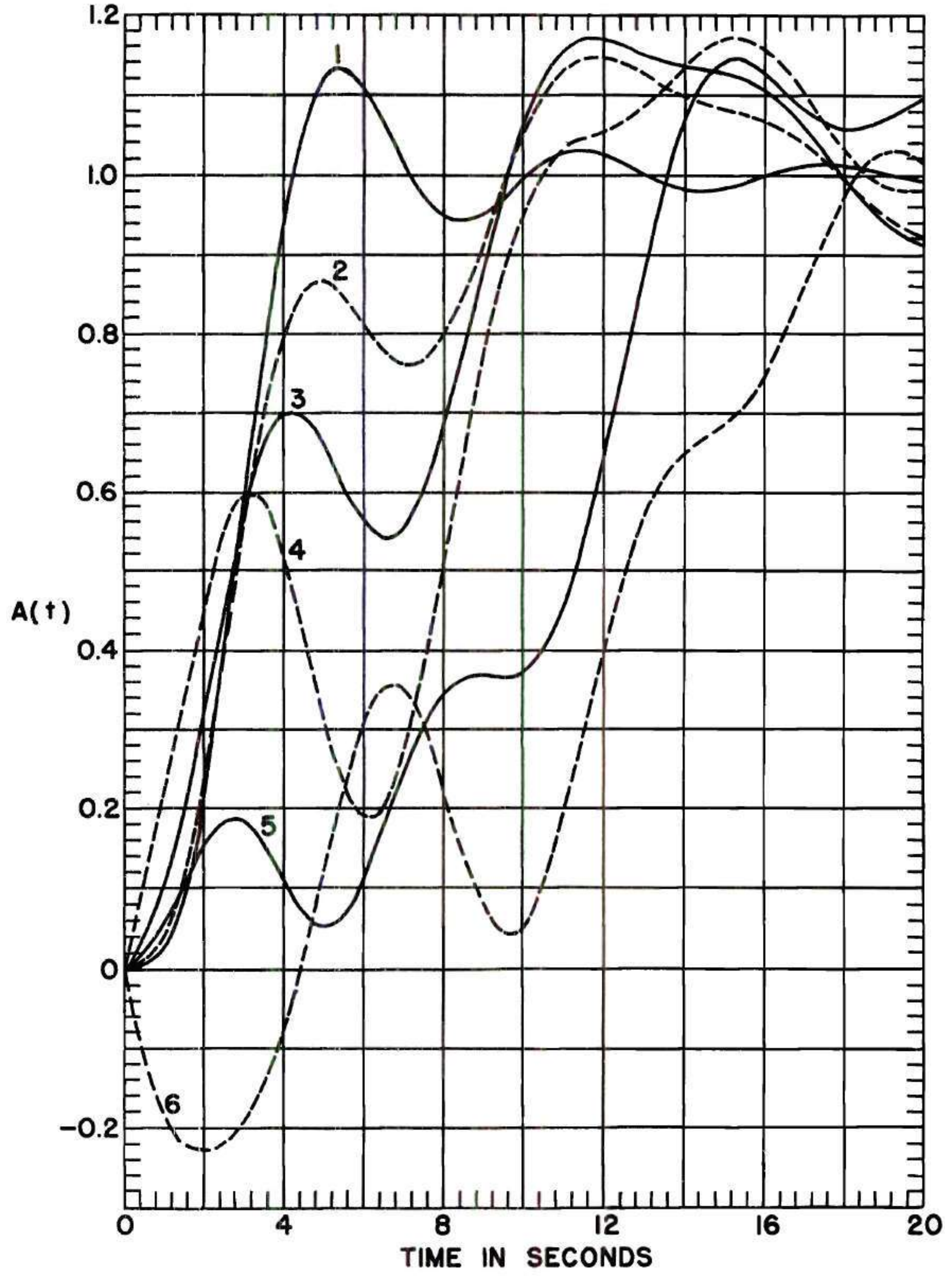

Figure 19. Unit Step Response for Case VIII

CHAPTER V

CONCLUSIONS AND DISCUSSION OF RESULTS

The work described in this study demonstrates the practicability of digital computer techniques in the study and determination of the phase and transient characteristics of networks. With the application of Hilbert and Fourier transforms, a method of computing, without difficulty, useful approximations to the phase and unit step response of low-pass networks was determined. The results presented in Chapter IV differ from the results presented by others who have studied low-pass networks only in the methods used to obtain these results.

It is well known that an immittance function can be realized or synthesized as a driving-point function only if the phase of the immittance function remains within the range $-\pi/2$ to $+\pi/2$ for all j-axis points. In Case II, we observe that if the slope of the attenuation beyond the pass band exceeds 20 decibels per decade, the phase falls outside of the range indicated above, and the network functions corresponding to the attenuation characteristics cannot be realized as driving-point functions. In Case IV, we do not have semi-infinite constant slopes beyond the pass bands, but attenuation of constant slope to flat stop bands. It is seen that the slope of attenuation can exceed 20 decibels per decade while the phase does not fall outside of the range indicated above, and the corresponding network functions can be realized as driving-point functions.

The most important characteristics of the response of a unit step function are the rise time and percentage overshoot. The rise time is the time required for the response to go from 10 per cent to 90 per cent of its final value. The percentage overshoot is the percentage of the final value by which the response exceeds its final value. Two other characteristics are sometimes also of importance, namely the delay time and the settling time. Delay time is the time required for the response to reach 50 per cent of its final value. Settling time is a measure of the time for the response to settle to within 5 per cent of its final value and not exceed this percentage later in the response.

The attenuation characteristics in Case I and Case II differ only by the 3 db filets at the cutoff frequency in the Butterworth attenuation characteristics of Case I. Comparing the step responses of Figures 5 and 6, and referring to Table 2, it is seen that when there is a 3 db filet at the cutoff frequency in the attenuation characteristic, the corresponding step response has less overshoot and a longer rise time. It is also seen that the delay time for Case I is longer, while the duration of the overshoot-oscillation is less.

In Case III, the slopes of the attenuation beyond the pass bands are integral multiples of 20 decibels per decade. It is seen in Table 3 that as the attenuation of the stop bands is increased there is a small change in the rise time of the step responses and only a maximum change of 2.4 per cent in the percentage overshoot. Table 4 shows a comparison of the characteristics of the step responses of Cases II and III which have equal slopes of attenuation beyond the pass bands in the corresponding attenuation characteristics. It can be seen from Table 4 that the step responses corresponding to attenuation characteristics having semi-infinite constant slopes beyond the pass bands have longer delay times and durations of overshoot-oscillation.

Table 5 shows a comparison of the characteristics of the step responses of Case IV. Figure 13 illustrates the effect on the phase and step response of a network function having an extremely sharp cutoff beyond the pass band. It is noted in Figure 13 that the delay times of the step responses exceed the rise times. Figures 14 to 17 show a large change in the step responses and a small change in the phase of the network functions having equal ripple in their pass bands and sloping pass bands. Figures 18 and 19 illustrate the effect on the phase and step response resulting from a sharp null in the pass band of a network function attenuation characteristic.

A P P E N D I C E S

APPENDIX I

DETERMINATION OF PHASE CHARACTERISTIC BY HILBERT TRANSFORMS

Two functions, $u(\sigma,\omega)$ and $v(\sigma,\omega)$ satisfying the Cauchy-Riemann equations are said to be conjugate potential functions. These two functions are the real and imaginary parts of an analytic function of the complex variable $\sigma + j\omega$. If

$$F(\sigma + j\omega) = u(\sigma,\omega) + j\, v(\sigma,\omega), \tag{10}$$

then $u(0,\omega) = u(\omega)$ and $v(0,\omega) = v(\omega)$ are related by the so-called Hilbert transforms

$$u(\omega) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{v(\lambda)}{\lambda - \omega}\, d\lambda + u(\ ) \tag{11}$$

and

$$v(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{u(\lambda)}{\lambda - \omega}\, d\lambda. \tag{12}$$

The logarithm of an immittance function is an analytic function of the complex variable $\sigma + j\omega$ whose real and imaginary parts are the magnitude and phase of the immittance. The immittance function evaluated for $\sigma = 0$ is written

$$H(0 + j\omega) = H(j\omega) = \left|H(j\omega)\right| e^{j\beta(\omega)}, \tag{13}$$

where $|H(j\omega)|$ and $\beta(\omega)$ are the magnitude and phase, respectively, of the immittance. Then

$$\log_e H(j\omega) = \log_e |H(j\omega)| + j\,\beta(\omega) = A(\omega) + j\,B(\omega), \tag{14}$$

where $A(\omega)$ and $B(\omega)$ are the attenuation in nepers and the phase in radians of the immittance function, respectively. From Equation 14 and Equations 11 and 12, we have

$$A(\omega) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{B(\lambda)}{\lambda - \omega}\,d\lambda + A(\infty) \tag{15}$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{A(\lambda)}{\lambda - \omega}\,d\lambda. \tag{16}$$

Since at real frequencies the real and imaginary parts of all the functions of interest are respectively even and odd functions of ω, we can write the alternative forms

$$A(\omega) = -\frac{2}{\pi}\int_{0}^{\infty}\frac{\lambda B(\lambda)}{\lambda^2 - \omega^2}\,d\lambda + A(\infty) \tag{17}$$

and

$$B(\omega) = \frac{2\omega}{\pi}\int_{0}^{\infty}\frac{A(\lambda)}{\lambda^2 - \omega^2}\,d\lambda. \tag{18}$$

The function $B(\omega)$ can be written (11) in terms of the derivative of $A(\omega)$ as

$$B(\omega) = -\frac{1}{\pi}\int_0^{\infty} \frac{dA(\lambda)}{d\lambda} \log_e\left|\frac{\lambda - \omega}{\lambda + \omega}\right| d\lambda. \tag{19}$$

An additional alternate form is obtained if we introduce a logarithmic frequency scale. If we let

$$u = \log_e \frac{\lambda}{\omega},$$

then $\lambda = \omega e^u$ and Equation 19 becomes

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dA(\omega e^u)}{du} \log_e \coth \left|\frac{u}{2}\right| du. \tag{20}$$

Now, consider an attenuation characteristic plotted versus the logarithm of frequency in which the attenuation is constant on one side of a prescribed frequency ω_0 and has a constant slope of one thereafter. For this attenuation characteristic the phase, $B(\omega_c)$, at any frequency ω_c is given by Equation 20 as

$$B(\omega_c) = \frac{1}{\pi}\int_{u_c}^{\infty} \log_e \coth \left|\frac{u}{2}\right| du, \tag{21}$$

where $u_c = \log_e \omega_0/\omega_c$, since in the range below u_c the slope is zero and the integrand in Equation 20 vanishes. For purposes of computations, Equation 21 can be rewritten using Equation 19 as

$$B(\omega_c) = \frac{1}{\pi}\int_0^{x_c} \log_e \left|\frac{1 + x}{1 - x}\right| \frac{dx}{x}, \tag{22}$$

where $x_c = \omega_c/\omega_0$. We can now evaluate $B(x_c)$ by a power series expansion.

If the series

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right), \; x^2 < 1, \tag{23}$$

is substituted into Equation 22 and integrated term by term, the result is

$$B(x_c) = \frac{2}{\pi}\left(x_c + \frac{x_c^3}{9} + \frac{x_c^5}{25} + \ldots\right), \; x_c < 1. \tag{24}$$

If $x_c = 1$, then $B(x_c) = \pi/4$, since the definite integral of Equation 22 with $x_c = 1$ is known to be $\pi^2/4$. The phase characteristic associated with the attenuation characteristic described above exhibits odd symmetry on a logarithmic frequency scale about the point ω_0. Hence, we have for $\omega_c > \omega_0$,

$$B(x_c) = \frac{\pi}{2} - B(x_c') \tag{25}$$

where $x_c' = \omega_0/\omega_c$ and $x_c' < 1$.

To obtain an approximation to Equation 24, consider the symmetrical transformation between x and y

$$y = \frac{1-x}{1+x} \qquad\qquad y_c = \frac{1-x_c}{1+x_c}\,. \tag{26}$$

In terms of the new variable y, Equation 22 can be written as

$$B(x_c) = -\frac{1}{\pi}\int_{x=0}^{x=x_c} \log_e y \; d(\log_e x). \tag{27}$$

Integrating by parts we have

$$B(x_c) = -\frac{1}{\pi}\left[\log_e x \log_e y\right]_{x=0}^{x=x_c} + \frac{1}{\pi}\int_{x=0}^{x=x_c} \log_e x \; d(\log_e y), \qquad (28)$$

which can be rewritten as

$$B(x_c) = -\frac{1}{\pi}\log_e x_c \log_e y_c - \frac{1}{\pi}\int_{y=y_c}^{y=1} \log_e x \; d(\log_e y). \qquad (29)$$

We may write the integral of Equation 29 as

$$\frac{1}{\pi}\int_{y=y_c}^{y=1} \log_e x \; d(\log_e y) = \frac{1}{\pi}\int_{y=0}^{y=1} \log_e x \; d(\log_e y) \qquad (30)$$

$$- \frac{1}{\pi}\int_{y=0}^{y=y_c} \log_e x \; d(\log_e y)$$

$$= -\frac{1}{\pi}\int_{y=0}^{y=1} \log_e \left|\frac{1+y}{1-y}\right| \frac{dy}{y} - \frac{1}{\pi}\int_{y=0}^{y=y_c} \log_e x \; d(\log_e y).$$

The integral from zero to one is equal to $\pi/4$ as previously stated, and the integral from zero to y_c is the phase characteristic at the point y_c from Equation 27. Equation 29 can now be written as

$$B(x_c) + B(y_c) = \frac{\pi}{4} - \frac{1}{\pi}\log_e x_c \log_e y_c, \qquad (31)$$

if x_c and y_c satisfy Equation 26.

In Equation 26, if x_c is near unity, the corresponding y_c is very small but it increases as x_c decreases and the two become equal at

$x_c = y_c \doteq 0.414$. Thus, the phase characteristic can be computed at all frequencies if we know it only between zero and 0.414. Within this region we can expect the power series expansion for $B(x_c)$ to converge rapidly.

If we use the first five terms of Equation 24 in conjunction with Equation 31, we have

$$B(x_c) \doteq \frac{2}{\pi}\left(x_c + \frac{x_c^3}{9} + \frac{x_c^5}{25} + \frac{x_c^7}{49} + \frac{x_c^9}{81}\right), \quad 0 \le x_c \le 0.414 \tag{32}$$

$$B(x_c) \doteq \frac{\pi}{4} - \frac{1}{\pi}\log_e(x_c)\log_e\left(\frac{1-x_c}{1+x_c}\right) - \frac{2}{\pi}\left[\left(\frac{1-x_c}{1+x_c}\right) + \frac{1}{9}\left(\frac{1-x_c}{1+x_c}\right)^3 + \frac{1}{25}\left(\frac{1-x_c}{1+x_c}\right)^5 + \frac{1}{49}\left(\frac{1-x_c}{1+x_c}\right)^7 + \frac{1}{81}\left(\frac{1-x_c}{1+x_c}\right)^9\right], \quad 0.414 \le x_c < 1. \tag{33}$$

APPENDIX II

THE DERIVATION OF THE FOURIER TRANSFORM RELATING THE UNIT STEP RESPONSE TO THE REAL PART OF AN IMMITTANCE FUNCTION

The Fourier integral can be written in various forms. For instance, we can write

$$f(t) = \int_{-\infty}^{+\infty} g(\omega)\, e^{j\omega t}\, d\omega \tag{34}$$

where

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt. \tag{35}$$

These two expressions constitute what is known as a Fourier transform pair. By use of Equation 35, a time function f(t) may be changed to its corresponding frequency spectrum g(ω). Similarly, by the use of Equation 34, a given frequency spectrum g(ω) may be changed into the corresponding time function f(t).

The direct Fourier transform does not converge for the unit step function but can be approached by the use of an exponentially decaying time function. Such a function is shown in Figure 20(a), and its direct Fourier transform is, from Equation 35,

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{0} \left(-\frac{1}{2}\, e^{at}\right) e^{-j\omega t}\, dt + \frac{1}{2\pi} \int_{0}^{+\infty} \left(\frac{1}{2}\, e^{-at}\right) e^{-j\omega t}\, dt \tag{36}$$

$$= \frac{\omega}{2\pi j\,(a^2 + \omega^2)}\,.$$

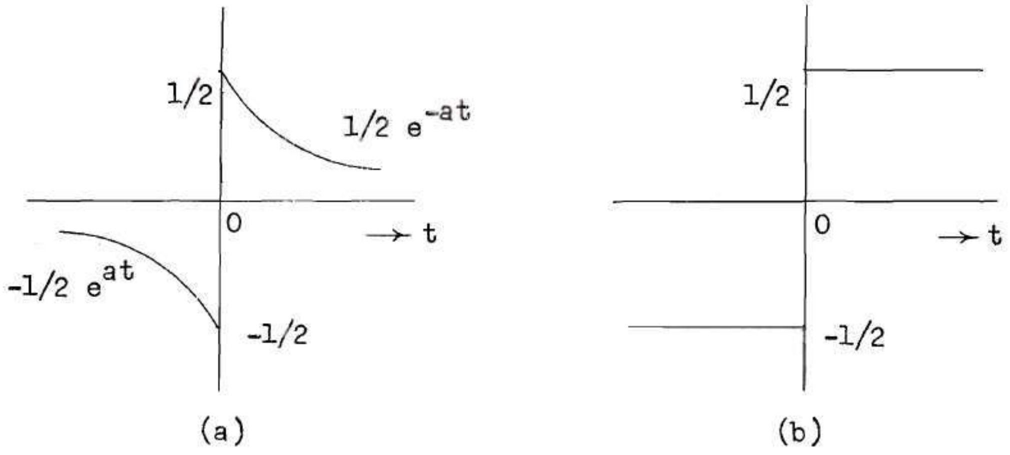

Figure 20. Exponentially Decaying and Step Functions

When a is allowed to approach zero, the frequency spectrum $g(\omega)$ approaches

$$g(\omega) = \frac{1}{2\pi j\omega} \tag{37}$$

and the resulting step wave is shown in Figure 20(b).

To make the usual unit step function $u_{-1}(t)$, which is zero when time $t < 0$ and unity when $t > 0$, a 1/2 must be added. Hence, from Equation 34 and Equation 37 we have

$$u_{-1}(t) = \frac{1}{2} + \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi j\omega}\right) e^{j\omega t}\, d\omega. \tag{38}$$

In the steady state case, the immittance of a network is a function of $j\omega$ and can be split into a real and imaginary part as

$$H(j\omega) = R(\omega) + j\, X(\omega). \tag{39}$$

$R(\omega)$ is called the real part of the immittance and is an even function of ω. $X(\omega)$ is called the imaginary part of the immittance and is an odd function of ω. By use of Equation 38 and Equation 39, the unit step response $A(t)$ of a unit step function is given by

$$A(t) = \frac{1}{2} R(0) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(R(\omega) + j\, X(\omega) \right) \frac{e^{j\omega t}}{j\omega}\, d\omega \qquad (40)$$

$$= \frac{1}{2} R(0)$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\left(R(\omega)\sin\omega t + X(\omega)\cos\omega t \right) + j \left(X(\omega)\sin\omega t - R(\omega)\cos\omega t \right)}{\omega}\, d\omega.$$

Now the imaginary part of the integrand,

$$\frac{X(\omega)\sin\omega t - R(\omega)\cos\omega t}{\omega} \qquad (41)$$

is an odd function of ω, and will vanish when integrated between the limits $-\infty$ and $+\infty$. The real part of the integrand is an even function of ω, and we can write

$$A(t) = \frac{1}{2} R(0) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{R(\omega)\sin\omega t + X(\omega)\cos\omega t}{\omega}\, d\omega \qquad (42)$$

$$= \frac{1}{2} R(0) + \frac{1}{\pi} \int_{0}^{\infty} \frac{R(\omega)\sin\omega t}{\omega}\, d\omega + \frac{1}{\pi} \int_{0}^{\infty} \frac{X(\omega)\cos\omega t}{\omega}\, d\omega.$$

For the response of Equation 42 to be physically realizable, it is necessary that $A(t)$ be zero for all values of time $t < 0$. Letting $t = -t'$ and setting Equation 42 equal to zero, we have

$$0 = \frac{1}{2} R(0) - \frac{1}{\pi} \int_0^{\infty} \frac{R(\omega)\sin\omega t'}{\omega} d\omega + \frac{1}{\pi} \int_0^{\infty} \frac{X(\omega)\cos\omega t'}{\omega} d\omega. \quad (43)$$

Now if the primes are dropped and Equation 43 is subtracted from Equation 42, the unit step response, $A(t)$, is given by

$$A(t) = \frac{1}{\pi} \int_0^{\infty} \frac{R(\omega)\sin\omega t}{\omega} d\omega, \ t \geq 0. \quad (44)$$

APPENDIX III

PROGRAM FOR BURROUGHS 220 DIGITAL COMPUTER

```
COMMENT  THIS PROGRAM CALCULATES THE PHASE AND UNIT STEP RESPONSE
  OF A LOW-PASS NETWORK FUNCTION FROM THE ATTENUATION CHARACTERISTIC
  OF THE NETWORK FUNCTION PLOTTED VERSUS THE LOG OF FREQUENCY $
COMMENT  THE FOLLOWING DATA IS REQUIRED FROM THE ATTENUATION
  CHARACTERISTIC.
    H - NUMBER OF CRITICAL POINTS OF STRAIGHT-LINE APPROXIMATION
        TO THE ATTENUATION CHARACTERISTIC
    ATO - ATTENUATION AT ZERO FREQUENCY
    WO(N) - CRITICAL FREQUENCY
    AO(N) - ATTENUATION AT CRITICAL FREQUENCY ON STRAIGHT-LINE
            APPROXIMATION
    AT(I) - ATTENUATION AT 19 EQUALLY-SPACED FREQUENCIES IN THE DECADE
            0.01 TO 0.1 AND 37 EQUALLY-SPACED FREQUENCIES IN THE DECADE
            0.1 TO 1.0
    ATX(P) - ATTENUATION AT 37 EQUALLY-SPACED FREQUENCIES IN THE DECADE
             1.0 TO 10.0 $
INTEGER I, N, H, J, P, II( ), FI( ) $
ARRAY WO(20), AO(20), K(20), THETA(20), ATX(37), W(249), AT(249),
  PHASE(249), M(249), RDW(249), II(3) = (14,33,70),
  FI(3) = (30,67,248), DW(3) = (0.005,0.025,0.05),
  WI(3) = (0.01,0.1,1.0) $
INPUT DATA1 (H, ATO, FOR N = (1,1,H) $ WO(N)),
  DATA2 (FOR N = (1,1,H) $ AO(N)),
  DATA3 (FOR I = (13,1,68) $ AT(I)),
  DATA4 (FOR P = (1,1,37) $ ATX(P)) $
OUTPUT OUT1 (W(I),(57.29578)(PHASE(I)),PHASE(I)), OUT2 (T,USR) $
FORMAT ATITLE (B25,*FREQUENCY*,B21,*PHASE*,W3,W2),
  BTITLE (B20,*RADIANS PER SECOND*,B7,*DEGREES*,B11,*RADIANS*,W2),
```

```
  CTITLE (B26,*TIME*,B14,*UNIT*,W3,W2),
  DTITLE (B25,*SECONDS*,B7,*STEP RESPONSE*,W2),
  FMT1 (B23,X10.7,B11,X10.4,B8,X10.6,W0),
  FMT2 (B26,X5.2,B12,X6.4,W0) $
FUNCTION BX(A,B) = (0.63661977)(A(1.0+B(0.11111111+B(0.04+B(0.02040816
  +(0.01234568)(B)))))) $
FUNCTION BZ(A,B,C) = (0.78539816)-(0.31830989)(LOG(A))(LOG(B))
  -(0.63661977)(B(1.0+C(0.11111111+C(0.04+C(0.02040816+(0.01234568)
  (C)))))) $
START.. READ($$DATA1) $ READ($$DATA2) $ READ($$DATA3) $ READ($$DATA4) $
FOR P = (1,1,36) $ BEGIN AT(69+(P-1)(5)) = ATX(P) $
  FOR I = ((70+(P-1)(5)),1,(73+(P-1)(5))) $
    BEGIN AT(I) = AT(I-1)+(0.2)(ATX(P+1)-ATX(P)) END END $
  AT(249) = ATX(37) $
FOR I = (1,1,12) $ BEGIN AT(I) = AT0 END $
FOR J = (1,1,4) $ BEGIN Q = (10.0)*-(7-J) $
  FOR I = ((3J-2),1,3J) $ BEGIN W(I) = Q+(I-(3J-2))(4.5)(Q) END END $
FOR J = (1,1,3) $
  BEGIN FOR I = ((II(J)-1),1,(FI(J)+1)) $
    BEGIN W(I) = WI(J)+(I-(II(J)-1))(DW(J)) END END $
FOR N = (2,1,H) $
  BEGIN K(N) = (AO(N)-AO(N-1))/((8.6858896)(LOG(WO(N)/WO(N-1)))) END $
WO(H) = WO(H)+1.0**46 $
FOR I = (1,1,249) $
  BEGIN FOR N = (1,1,H) $
    BEGIN EITHER IF W(I) GTR WO(N) $ GO TO ALT2 $
    OR IF W(I) EQL WO(N) $ GO TO ALT1 $
    OTHERWISE $ BEGIN X = W(I)/WO(N) $ X2 = XX END $
  EITHER IF X GTR 0.414 $
    BEGIN Y = (1.0-X)/(1.0+X) $ Y2 = YY $
    THETA(N) = BZ(X,Y,Y2) $ GO TO ALT3 END $
  OTHERWISE $ BEGIN THETA(N) = BX(X,X2) $ GO TO ALT3 END $
  ALT1.. THETA(N) = 0.78539816 $ GO TO ALT3 $
  ALT2.. X = WO(N)/W(I) $ X2 = XX $
```

```
  EITHER IF X GTR 0.414 $
    BEGIN Y = (1.0-X)/(1.0+X) $ Y2 = YY $
    THETA(N) = 1.5707963-BZ(X,Y,Y2) $ GO TO ALT3 END $
  OTHERWISE $ BEGIN THETA(N) = 1.5707963-BX(X,X2) $ GO TO ALT3 END $
  ALT3.. N = N END $
  PHASE(I) = 0.0 $
  FOR N = (2,1,H) $
    BEGIN PHASE(I) = PHASE(I)+(K(N))(THETA(N-1)-THETA(N)) END $
  EITHER IF AT(I) EQL 0.0 $
    BEGIN RDW(I) = ((0.21220659)(COS(PHASE(I))))/W(I) END $
  OR IF AT(I) NEQ AT(I-1) $
    BEGIN M(I) = (10.0)*((0.05)(AT(I))) $
    RDW(I) = ((M(I))((0.21220659)(COS(PHASE(I)))))/W(I) END $
  OTHERWISE $ BEGIN M(I) = M(I-1) $
    RDW(I) = ((M(I))((0.21220659)(COS(PHASE(I)))))/W(I) END $
  IF ( (I EQL 1) OR (I EQL 53) OR (I EQL 105) OR (I EQL 157)
    OR (I EQL 209) ) $
    BEGIN WRITE($$ATITLE) $ WRITE($$BTITLE) END $
  WRITE($$OUT1,FMT1) END $
WRITE($$CTITLE) $ WRITE($$DTITLE) $
FOR T = (0.0,0.4,20.0) $
  BEGIN USR = 0.0 $ FOR J = (1,1,4) $ BEGIN Q = (10.0)*-(7-J) $
    USR = USR+((4.5)(Q))(RDW(3J-2)(SIN(W(3J-2).T))+4.0(RDW(3J-1)
    (SIN(W(3J-1).T)))+2.0(RDW(3J)(SIN(W(3J).T))) END $
  FOR J = (1,1,3) $
    BEGIN YY2 = 0.0 $ YY4 = 0.0 $
    S20 = SIN((2.0)(DW(J))(T)) $ C20 = COS((2.0)(DW(J))(T)) $
    S1 = SIN((WI(J)-DW(J))(T)) $ C1 = COS((WI(J)-DW(J))(T)) $
    S2 = SIN((WI(J))(T)) $ C2 = COS((WI(J))(T)) $
    FOR I = (II(J),2,FI(J)) $
     BEGIN SS1 = S1 $ CC1 = C1 $ SS2 = S2 $ CC2 = C2 $
     S1 = (SS1)(C20)+(CC1)(S20) $ C1 = (CC1)(C20)-(SS1)(S20) $
     S2 = (SS2)(C20)+(CC2)(S20) $ C2 = (CC2)(C20)-(SS2)(S20) $
     YY2 = YY2+RDW(I+1)(S2) $ YY4 = YY4+RDW(I)(S1) END $
```

```
    USR = USR+(DW(J))(RDW(II(J)+1)(SIN(WI(J).T))+2.0(YY2)+4.0(YY4)
       -(RDW(FI(J)+1)(S2))) END $
   WRITE($$OUT2,FMT2) END $
GO TO START $ FINISH $
```

SAMPLE SET OF DATA

```
9 0.0 .5 .6 .7 .8 .9 1.0 1.2 2.0 10.0
0.0 -.25 -.5 -1.0 -1.75 -3.0 -6.0 -18.0 -60.0
0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
0.0
0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 -.025 -.05
-.085 -.11 -.15 -.2 -.275 -.35 -.425 -.5 -.625 -.75 -.875 -1.0 -1.2 -1.4
-1.625 -1.85 -2.1 -2.4 -2.7 -3.0
-3.0 -6.85 -10.95 -14.75 -18.5 -21.15 -23.85 -26.25 -28.5 -30.6 -32.55
-34.35 -36.15 -37.65 -39.15 -40.5 -41.85 -43.2 -44.4 -45.6 -46.65 -47.7
-48.75 -49.65 -50.55 -51.45 -52.35 -53.25 -54.15 -54.9 -55.65 -56.4
-57.15 -57.9 -58.65 -59.4 -60.0
```

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